On the Relationships between Period and Cohort Fertility

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1. Introduction

The relationships between period and cohort fertility measures have been one of the central issues in demography. In this regard, Ryder (1960, 1964) was one the first pioneers in trying to establish the relationships mathematically, who also coined the term “demographic translation” for the process. Since then, many researchers have made contributions in terms of improving and extending the original translation equations developed by Ryder (e.g. Foster, 1990; Ní Bhrolcháin, 1992; Bongaarts and Feeney, 1998; Keilman, 2000; Kohler and Philipov, 2001; Zeng and Land, 2002; Rodríguez, 2006), and a number of important results have been achieved.

However, there has been, to some extent, a lack of a unified analytical framework, which connects the dots in a more systematic way. This paper attempts to further examine the quantitative relationships between period and cohort fertility based on a unified analytical framework.

2. A general relationship between period and cohort fertility

In demography, age, period and cohort are three key dimensions and the Lexis diagram is a powerful tool to facilitate age-period-cohort (A-P-C) analysis. The Lexis diagram provides a graphical representation of the relationships among age, period and cohort. Figure 1 shows a portion of the Lexis diagram, in which we have the following correspondences:

- **Cohort-age analysis** (i.e. cohort \( y \) and age \( a \)) corresponds to parallelogram \( DGHE \), which crosscuts two years (i.e. \( t-1 \) and \( t \)) and is called the cohort-age parallelogram.
- **Period-cohort analysis** (i.e. year \( t \) and cohort \( y \)) corresponds to parallelogram \( DHEA \), which crosscuts two ages (i.e. \( a \) and \( a+1 \)) and is called the period-cohort parallelogram.
- **Period-age analysis** (i.e. year \( t \) and age \( a \)) corresponds to square \( DHIE \), which crosscuts two cohorts (i.e. cohort \( y \) and cohort \( y+1 \)) and is called the period-age square.
- **Age-period-cohort analysis** (i.e. age \( a \), year \( t \), and cohort \( y \)) corresponds to triangle \( DHE \), which is called the age-period-cohort triangle.

When discussing measures of demographic events (e.g. fertility), it is very important to be clear about which geometric shape is addressed. For example, the conventional (period) total fertility rate (TFR)
is defined as the sum (total) of the age-specific fertility rates based on the squares (i.e. the period-age squares) of the corresponding year.

Figure 1. Lexis diagram for age-period-cohort analysis

Since the focus of this paper is on the relationships between period and cohort fertility, the geometric shape to be used is the period-cohort parallelogram (i.e. $DHEA$ in Figure 1). Therefore, for the purpose of this paper, the Lexis diagram in Figure 2 is used as a unified analytical framework for examining the quantitative relationships between period and cohort fertility. When Figure 2 is read horizontally, it relates to age analysis (i.e. age-period, or age-cohort); when Figure 2 is read vertically, it relates to period analysis (i.e. period-age, or period-cohort); when Figure 2 is read diagonally, it relates to cohort analysis (i.e. cohort-age, or cohort-period).

Suppose that year $t$ is the period under study. From Figure 2, it is obvious that women who are aged $a$ at the beginning of year $t$ must be born in year $t - a - 1$. Now we define a few variables as follows.

Let $W^c_y$ represent the number of women who were born in year $y$ ($y = t-50, t-49, \ldots, t-16$), where the superscript $c$ stands for cohort. Obviously, $W^c_y$ constitute a birth cohort. For women of birth cohort $y$ (i.e. diagonal in Figure 2), let $W^c_y(a)$ represent the number of women who are aged $a$ ($a = 15, 16, \ldots, 49$) at the beginning of the corresponding year. We assume that there is no mortality before the end of women’s reproductive lifespan, then we have $W^c_y(15) = W^c_y(16) = \cdots = W^c_y(49) = W^c_y$.

1 Please note that in this paper, age $a$ always refers to the age of women at the beginning of a corresponding year.
### Figure 2. Lexis diagram – A unified analytical framework

<table>
<thead>
<tr>
<th>Calendar year</th>
<th>Age</th>
<th>t-50</th>
<th>t-(a+1)</th>
<th>t-17</th>
<th>t-16</th>
</tr>
</thead>
<tbody>
<tr>
<td>49</td>
<td>50</td>
<td></td>
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<td>...</td>
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<tr>
<td>16</td>
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<td></td>
</tr>
<tr>
<td>15</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Birth cohort</td>
<td>(year of birth)</td>
<td>t-50</td>
<td>t-(a+1)</td>
<td>t-17</td>
<td>t-16</td>
</tr>
</tbody>
</table>
Let $B^c_y$ denote the total number of live births that $W^c_y$ delivered during their entire reproductive lifespan and $B^c_y(a)$ denote the number of live births that $W^c_y$ delivered during the year at the beginning of which the women were aged $a$. Here, $B^c_y(a)$ corresponds to the parallelogram concerned. It is obvious that $B^c_y = \sum_{a=15}^{49} B^c_y(a)$.

For birth cohort $y$, we define its (cohort) age-specific fertility rates as follows:

$$f^c_y(a) = B^c_y(a)/W^c_y(a), \ a = 15, 16, \ldots, 49$$

(2.1)

Please note that the $f^c_y(a)$ defined above are actually cohort-period measures, i.e. they are based on the cohort-period parallelogram (i.e. DHEA in Figure 1), not the cohort-age parallelogram (i.e. DGHE in Figure 1).

For birth cohort $y$, we define its (cohort) lifetime fertility rate (LFR) as the average number of live births that $W^c_y$ delivered during their entire reproductive lifespan, i.e. $LFR_y = B^c_y/W^c_y$. Then, we have

$$LFR_y = \frac{\sum_{a=15}^{49} B^c_y(a)}{W^c_y} = \frac{\sum_{a=15}^{49} B^c_y(a)}{W^c_y(a)} = \frac{\sum_{a=15}^{49} B^c_y(a)}{W^c_y(a)} = \sum_{a=15}^{49} f^c_y(a)$$

(2.2)

Equation (2.2) indicates that, for each birth cohort ($y$), its lifetime fertility rate equals the sum of the cohort age-specific fertility rates.

For birth cohort $y$, the sequence $\{f^c_y(a)\mid a = 15, 16, \ldots, 49\}$ constitutes an age distribution of the lifetime fertility rate of the cohort (i.e. $LFR_y$). For the sequence $\{f^c_y(a)\mid a = 15, 16, \ldots, 49\}$, we define its standardized age pattern (schedule) of fertility as $\{h^c_y(a)\mid a = 15, 16, \ldots, 49\}$, where $h^c_y(a) = f^c_y(a)/LFR_y, \ a = 15, 16, \ldots, 49$. It is obvious that $h^c_y(a) \geq 0 \ (a = 15, 16, \ldots, 49)$ and $\sum_{a=15}^{49} h^c_y(a) = 1$. From the definition above, we see that for birth cohort $y$, $h^c_y(a)$% of the $LFR_y$ was born in year $y+a+1$, at the beginning of which, the women were aged $a$. From the above definitions, we obtain $h^c_y(a) = B^c_y(a)/B^c_y, \ a = 15, 16, \ldots, 49$. Furthermore, we have
Equation (2.3) indicates that \( f^c_{y}(a) \) can be expressed as the product of two cohort factors, one is a cohort fertility level factor (i.e. \( LFR_y \)), the other is a cohort fertility timing factor (i.e. \( h^c_{y}(a) \)). Therefore, any cohort or period fertility measures based on \( f^c_{y}(a) \) will have a level component and a timing component.

Now let’s look at the Lexis diagram in Figure 2 from a period perspective. It is obvious that, for year \( t \), we have two period curves of age-specific fertility rates, i.e. \( \{f^c_{t-(a+1)}(a) \mid a = 15, 16, \ldots, 49\} \) and \( \{h^c_{t-(a+1)}(a) \mid a = 15, 16, \ldots, 49\} \). For convenience hereafter, we denote \( f^p_t(a) = f^c_{t-(a+1)}(a) \) and \( h^p_t(a) = h^c_{t-(a+1)}(a), a = 15, 16, \ldots, 49 \), where the superscript \( p \) stands for period. Then we have

\[
f^p_{t}(a) = h^p_{t}(a) \cdot LFR_{t-(a+1)}, \ a = 15, 16, \ldots, 49
\] (2.4)

Now, we define the (period) total fertility rate (TFR) for year \( t \) as follows:

\[
TFR_t = f^c_{t-16}(15) + f^c_{t-17}(16) + \ldots + f^c_{t-50}(49) = \sum_{a=15}^{49} f^c_{t-(a+1)}(a) = \sum_{a=15}^{49} f^p_{t}(a)
\] (2.5)

Please note that the definition of the total fertility rate of year \( t \) (i.e. \( TFR_t \)) above is based on the age-specific fertility rates that correspond to the concerned cohort-period parallelograms, while the conventional \( TFR \) is based on the age-specific fertility rates that correspond to the concerned period-age squares.

From equations (2.4) and (2.5), we have

\[
TFR_t = \sum_{a=15}^{49} f^p_{t}(a) = \sum_{a=15}^{49} [h^p_{t}(a) \cdot LFR_{t-(a+1)}]
\] (2.6)

Define \( G_t = \sum_{a=15}^{49} h^p_{t}(a) \), then equation (2.6) can be rewritten as

\[
TFR_t = G_t \cdot \sum_{a=15}^{49} \left( \frac{h^p_{t}(a) \cdot LFR_{t-(a+1)}}{G_t} \right) = G_t \cdot LFR_t
\] (2.7)

where \( G_t \) is a period (year \( t \)) adjustment factor, while \( LFR_t = \sum_{a=15}^{49} \left( \frac{h^p_{t}(a) \cdot LFR_{t-(a+1)}}{G_t} \right) \) is a

\[\text{Page 5}\]
weighted average of the concerned (cohort) lifetime fertility rates. Equation (2.7) shows that, under the assumption of no mortality before the end of the reproductive lifespan, the (period) $TFR$ in year $t$ (as defined in equation (2.5)) can be decomposed into two components, i.e. a level (quantum) factor (i.e. $LFRT$) and a timing (tempo) factor (i.e. $G_t$).

Mathematically, equation (2.7) provides a general expression for the quantitative relationship between the (period) total fertility rate and the corresponding (cohort) lifetime fertility rates. Butz and Ward (1979) noticed the relationship expressed in equation (2.7) and called the quantity $G_t$ the timing index (TI), and the quantity $LFRT$ the average completed fertility (AC).

Equation (2.7) also provides a way for decomposing the change in the (period) total fertility rate into different factors as follows:

$$TFR_{t+1} - TFR_t = G_{t+1} \cdot LFRT_{t+1} - G_t \cdot LFRT_t = E_{\text{level}} + E_{\text{timing}} + I \quad (2.8)$$

where $E_{\text{level}} = G_{t+1} \cdot (LFRT_{t+1} - LFRT_t)$ is the net effect of the change in the fertility quantum component (i.e. assuming that the fertility tempo component remains unchanged from year $t$ to year $t+1$); $E_{\text{timing}} = (G_{t+1} - G_t) \cdot LFRT_t$ is the net effect of the change in the fertility tempo component (i.e. assuming that the fertility quantum component remains unchanged from year $t$ to year $t+1$); and $I = (G_{t+1} - G_t) \cdot (LFRT_{t+1} - LFRT_t)$ is an interaction term, which reflects the joint effect of the simultaneous changes in both the fertility tempo and the fertility quantum components.

By its definition, the period quantity $G_t$ is affected by the childbearing behaviors of the concerned birth cohorts (i.e. $t-50, t-49, \ldots, t-16$) in year $t$. Theoretically, $G_t$ can take numerical values between 0 and 35. If all women of all the birth cohorts (i.e. $t-50, t-49, \ldots, t-16$) do not give any births in year $t$, then $G_t = 0$ (because in this case, we have $h^c_{t-16}(15) = h^c_{t-17}(16) = \cdots = h^c_{t-50}(49) = 0$). If all women of all the birth cohorts (i.e. $t-50, t-49, \ldots, t-16$) deliver all their (lifetime) births in year $t$, then $G_t = 35$ (because in this case, we have $h^c_{t-16}(15) = h^c_{t-17}(16) = \cdots = h^c_{t-50}(49) = 1$). Obviously, the above situations are two extremes. In reality, the numerical values of $G_t$ usually fall between 0.5 and 1.5. It is also obvious that if all the birth cohorts of women (i.e. $t-50, t-49, \ldots, t-16$) follow the same standardized cohort age pattern of fertility, then $G_t = 1$. 

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Let \( H_i^p(a) = k_i^p(a) G_i \), \( a = 15, 16, \ldots, 49 \), then it is obvious that \( H_i^p(a) \geq 0 \) and \( \sum_{a=15}^{49} H_i^p(a) = 1 \).

Therefore, \( \{ H_i^p(a) \mid a = 15, 16, \ldots, 49 \} \) constitutes a standardized period (year \( t \)) age pattern (schedule). Thus, equation (2.7) can be rewritten as

\[
TFR_t = G_t \cdot \sum_{a=15}^{49} (H_i^p(a) \cdot LFR_{t-(a+1)})
\]

(2.9)

Suppose that \( LFR_y \) can be expressed by the following \( n \)-degree polynomial of \( y \):

\[
LFR_y = \lambda_0 + \lambda_1 \cdot y + \lambda_2 \cdot y^2 + \cdots + \lambda_n \cdot y^n = \sum_{i=0}^{n} (\lambda_i \cdot y^i)
\]

(2.10)

where \( n \) is a non-negative integer and \( \lambda_i, i = 0, 1, 2, \ldots, n \), are the polynomial coefficients. Let \( T = t - 1 \), then from equations (2.9) and (2.10), we have

\[
TFR_t = G_t \cdot \sum_{a=15}^{49} H_i^p(a) \cdot \sum_{i=0}^{n} (\lambda_i \cdot (T - a)^i) = G_t \cdot \sum_{i=0}^{n} \left( \lambda_i \cdot \sum_{a=15}^{49} ((T - a)^i \cdot H_i^p(a)) \right)
\]

(2.11)

Let \( Q_n = \sum_{i=0}^{n} \left( \lambda_i \cdot \sum_{a=15}^{49} ((T - a)^i \cdot H_i^p(a)) \right) \), then equation (2.11) becomes

\[
TFR_t = G_t \cdot Q_n
\]

(2.12)

In other words, under the polynomial assumption about \( LFR_y \), we have \( \overline{LFR_t} = Q_n \).

By the binomial theorem, we have

\[
(T - a)^j = \sum_{i=0}^{j} \left[ \frac{j!}{j!(i-j)!} \cdot T^{i-j} \cdot (-a)^j \right] = \sum_{i=0}^{j} \left[ (-1)^j \cdot \left( \frac{j!}{j!(i-j)!} \right) \cdot T^{i-j} \cdot a^j \right]
\]

(2.13)

Therefore, we obtain

\[
Q_n = \sum_{i=0}^{n} \left\{ \lambda_i \cdot \sum_{a=15}^{49} (T - a)^j \cdot H_i^p(a) \right\}
\]

\[
= \sum_{i=0}^{n} \left\{ \lambda_i \cdot \sum_{a=15}^{49} \left[ (-1)^j \cdot \left( \frac{j!}{j!(i-j)!} \right) \cdot T^{i-j} \cdot a^j \cdot H_i^p(a) \right] \right\}
\]

\[
= \sum_{i=0}^{n} \left\{ \lambda_i \cdot \sum_{j=0}^{i} \left[ (-1)^j \cdot \left( \frac{j!}{j!(i-j)!} \right) \cdot T^{i-j} \cdot \sum_{a=15}^{49} (a^j \cdot H_i^p(a)) \right] \right\}
\]

(2.14)
To further explore the period quantity $Q_n$, we need to examine the period curve $H_t^p(a)$. In this connection, the (statistical) moments are important measures for describing $H_t^p(a)$. There are two types of moments for describing a probability distribution, i.e. the absolute moments (about zero or origin) and the central moments (about the mean). Suppose that function $p(x)$ represents a probability distribution (i.e. $p(x)$ satisfies $p(x) \geq 0$ and $\sum_{x} p(x) = 1$), then its moments are defined as follows:

The $r$th absolute moment (about zero or origin) of $p(x)$ is defined as

$$
\hat{M}_r(p) = \sum_{x} \left[ x^r \cdot p(x) \right]
$$

(2.15)

where $r$ is a non-negative integer. It is obvious that $\hat{M}_0(p) = 1$ and $\hat{M}_1(p) = \mu(p)$, where $\mu(p) = \sum_{x} [x \cdot p(x)]$ is the mean of $p(x)$.

The $r$th central moment (about the mean) of $p(x)$ is defined as

$$
\tilde{M}_r(p) = \sum_{x} \left[ (x - \mu(p))^r \cdot p(x) \right]
$$

(2.16)

where $r$ is a non-negative integer. It is obvious that $\tilde{M}_0(p) = 1$, $\tilde{M}_1(p) = 0$, and $\tilde{M}_2(p) = \nu(p)$, where $\nu(p) = \sum_{x} [(x - \mu(p))^2 \cdot p(x)]$ is the variance of $p(x)$.

Other relevant properties of the moments are given in Annex A.

From equation (2.14), we have

$$
Q_n = \sum_{j=0}^{n} \left\{ \lambda_j \cdot \sum_{i=0}^{d} \left[ (-1)^i \cdot \left( \frac{d!}{j!(i-j)!} \right) \cdot T^{i-j} \cdot \hat{M}_j(H_t^p) \right] \right\}
$$

(2.17)

It is obvious from equation (2.17) that

$$
Q_0 = \lambda_0 \cdot \hat{M}_0(H_t^p) = \lambda_0
$$

$$
Q_d = Q_{d-1} + \lambda_d \cdot \sum_{j=0}^{d} \left[ (-1)^j \cdot \left( \frac{d!}{j!(d-j)!} \right) \cdot T^{d-j} \cdot \hat{M}_j(H_t^p) \right], \quad d = 1, 2, \ldots, n
$$

(2.18)
Once the polynomial coefficients \( \lambda_i, \ i = 0, 1, 2, \ldots, n \), are known, equation (2.17) shows that \( Q_n \) is a linear function of the absolute moments of \( H^p_t(a) \), i.e. \( Q_n \) can be written in the following form

\[
Q_n = \lambda_0 + \sum_{i=1}^{n} (\bar{\lambda}_i \cdot \bar{M}_i(H^p_t))
\]  

Equation (2.19) shows that \( Q_n \) is determined by the polynomial coefficients and the absolute moments of the period curve \( H^p_t(a) \).

Based on the general relationship, expressed in equation (2.12), we will explore some specific relationships between the (period) total fertility rate and the (cohort) lifetime fertility rates. In this regard, we will look at the period level component (i.e. \( Q_n \)) and the period timing component (i.e. \( G_t \)) separately, as the two components may be considered “independent of each other” from a mathematical point of view.

3. Some specific expressions of \( Q_n \)

3.1 The (cohort) lifetime fertility rate remains constant over time (birth cohort)

Under this assumption, we have \( LFR_{y-16} = LFR_{y-17} = \cdots = LFR_{y-50} \), denoted as \( LFR \). This is equivalent to taking \( n = 0 \) in equation (2.10). Therefore, \( LFR_y = LFR = \lambda_0 \). Then from equation (2.18), we have \( Q_0 = \lambda_0 = LFR \). In this case, we have

\[
TFR_t = G_t \cdot Q_0 = G_t \cdot LFR
\]  

Equation (3.1) shows that even if the level of cohort fertility (i.e. lifetime fertility rate) is invariant over time (cohort), the (period) total fertility rate may be greater than, equal to, or smaller than the (cohort) lifetime fertility rate depending on the period adjustment factor for year \( t \) (i.e. \( G_t \)). If \( G_t = 1 \) (women procreate in year \( t \) “normally”), then we have \( TFR_t = LFR \). If \( G_t > 1 \) (women “favor” year \( t \) in terms of childbearing), then we have \( TFR_t > LFR \). If \( G_t < 1 \) (women “avoid” year \( t \) in terms of childbearing), then we have \( TFR_t < LFR \).

3.2 The (cohort) lifetime fertility rate changes linearly with time (birth cohort)

This is equivalent to taking \( n = 1 \) in equation (2.10), i.e. \( LFR_y = \lambda_0 + \lambda_1 \cdot y \). Then from equation (2.18), we have
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\[ Q_t = Q_0 + \lambda_1 \cdot (T - \hat{M}_t(H^p_t)) = \lambda_0 + \lambda_1 \cdot (T - \hat{M}_t(H^p_t)) = LFR_{t-M_t(H^p_t)} = LFR_{t-(\mu(H^p_t)+1)} \]  

(3.2)

In equation (3.2), \( LFR_{t-(\mu(H^p_t)+1)} \) is the lifetime fertility rate of birth cohort \( t-(\mu(H^p_t)+1) \), which is aged \( \mu(H^p_t) \) at the beginning of year \( t \). Figure 3 shows the relationship between \( Q_t \) and \( \mu(H^p_t) \).

Figure 3. Relationship between \( Q_t \) and \( \mu(H^p_t) \).

Equation (3.2) shows that under the linear assumption, \( Q_t \) is affected by the mean of the period curve \( H^p_t(a) \), but not affected by its shape (e.g. variance, skewness, kurtosis).

In this case, we have

\[ TFR_t = G_t \cdot Q_t = G_t \cdot LFR_{t-(\mu(H^p_t)+1)} \]

(3.3)

Equation (3.3) indicates that, under the assumption stated above, the mean age of the standardized period fertility curve \( H^p_t(a) \) (i.e. \( \mu(H^p_t) \)) plays a key role in determining the total fertility rate for year \( t \) (i.e. \( TFR_t \)).

Based on the data from China’s 2‰ fertility survey conducted in 1988, we produced Figure 4, which shows that the (cohort) lifetime fertility rates \( (LFR_y) \) of Chinese women born during 1931-1950 declined almost linearly with time (cohort), with the coefficient of determination being \( R^2 = 0.99 \).
In producing Figure 4, $\mu(H_t^p)$ was set at a constant of 29 years of age.

Figure 4. The values of LFR, TFR, and $G_t$ - China

3.3 The (cohort) lifetime fertility rate changes quadratically with time (birth cohort)

This is equivalent to taking $n = 2$ in equation (2.10), i.e. $LFR_y = \lambda_0 + \lambda_1 \cdot y + \lambda_2 \cdot y^2$. Therefore from equation (2.18), we have

$$Q_2 = Q_1 + \lambda_2 \cdot [T^2 - 2 \cdot T \cdot \hat{M}_t(H_t^p) + \hat{M}_2(H_t^p)]$$

$$= Q_1 + \lambda_2 \cdot [(T - \hat{M}_1(H_t^p))^2 + \hat{M}_2(H_t^p) - (\hat{M}_1(H_t^p))^2]$$

$$= \lambda_0 + \lambda_1 \cdot (T - \hat{M}_1(H_t^p)) + \lambda_2 \cdot (T - \hat{M}_1(H_t^p))^2$$

$$+ \lambda_2 \cdot [\hat{M}_2(H_t^p) - (\hat{M}_1(H_t^p))^2]$$

$$= LFR_{t-M_t(H_t^p)} + \lambda_2 \cdot \hat{D}_2(H_t^p)$$

$$= LFR_{t-(\mu(H_t^p)+1)} + \lambda_2 \cdot (\sigma(H_t^p))^2 \quad (3.4)$$

In equation (3.4), $LFR_{t-(\mu(H_t^p)+1)}$ is the lifetime fertility rate of birth cohort $t - (\mu(H_t^p)+1)$ (which
is aged $\mu(\hat{H}_t^p)$ at the beginning of year $t$, and $\lambda_2 \cdot (\sigma(\hat{H}_t^p))^2$ is a modification term. It is obvious that (i) when $\lambda_2 > 0$ (i.e. the parabola opens upwards), the modification term is positive, and (ii) when $\lambda_2 < 0$ (i.e. the parabola opens downwards), the modification term is negative. Equation (3.4) also shows that under the quadratic assumption, $Q_2$ is not only affected by the mean of the period curve $H_t^p(a)$, but also affected by its standard deviation. In this case, we have

$$TFR_t = G_t \cdot Q_2 = G_t \cdot \left[ LFR_{-\mu(\hat{H}_t^p)+1} + \lambda_2 \cdot (\sigma(\hat{H}_t^p))^2 \right]$$

(3.5)

3.4 The (cohort) lifetime fertility rate changes cubically with time (birth cohort)

This is equivalent to taking $n = 3$ in equation (2.10), i.e. $LFR_y = \lambda_0 + \lambda_1 \cdot y + \lambda_2 \cdot y^2 + \lambda_3 \cdot y^3$.

Therefore from equation (2.18), we have

$$Q_3 = Q_2 + \lambda_3 \cdot [T^3 - 3 \cdot T^2 \cdot \hat{M}_1(\hat{H}_t^p) + 3 \cdot T \cdot \hat{M}_2(\hat{H}_t^p) - \hat{M}_3(\hat{H}_t^p)]]$$

$$= \lambda_0 + \lambda_1 \cdot (T - \hat{M}_1(\hat{H}_t^p)) + \lambda_2 \cdot (T - \hat{M}_1(\hat{H}_t^p))^2 + \lambda_3 \cdot (T - \hat{M}_1(\hat{H}_t^p))^3$$

$$+ \lambda_2 \cdot \hat{D}_2(\hat{H}_t^p)$$

$$+ \lambda_3 \cdot (3 \cdot T \cdot \hat{D}_2(\hat{H}_t^p) - \hat{D}_3(\hat{H}_t^p))$$

$$= LFR_{-\mu(\hat{H}_t^p)+1} + \lambda_2 \cdot \hat{D}_2(\hat{H}_t^p) + \lambda_3 \cdot (3 \cdot T \cdot \hat{D}_2(\hat{H}_t^p) - \hat{D}_3(\hat{H}_t^p))$$

(3.6)

Since

$$3 \cdot T \cdot \hat{D}_2(\hat{H}_t^p) - \hat{D}_3(\hat{H}_t^p) = (\sigma(\hat{H}_t^p))^2 \cdot \omega_3$$

where $\omega_3 = 3 \cdot T - 3 \cdot \mu(\hat{H}_t^p) - \sigma(\hat{H}_t^p) \cdot \mu(\hat{H}_t^p)$, equation (3.6) can be rewritten as

$$Q_3 = LFR_{-\mu(\hat{H}_t^p)+1} + \lambda_2 \cdot (\sigma(\hat{H}_t^p))^2 + \lambda_2 \cdot (\sigma(\hat{H}_t^p))^2 \cdot \omega_3$$

$$= LFR_{-\mu(\hat{H}_t^p)+1} + (\lambda_2 + \lambda_3 \cdot \omega_3) \cdot (\sigma(\hat{H}_t^p))^2$$

(3.7)

In this case, we have

$$TFR_t = G_t \cdot Q_3 = G_t \cdot \left[ LFR_{-\mu(\hat{H}_t^p)+1} + (\lambda_2 + \lambda_3 \cdot \omega_3) \cdot (\sigma(\hat{H}_t^p))^2 \right]$$

(3.8)

3.5 The (cohort) lifetime fertility rate changes quartically with time (birth cohort)

This is equivalent to taking $n = 4$ in equation (2.10), i.e.
\[ LFR_t = \lambda_0 + \lambda_1 \cdot y + \lambda_2 \cdot y^2 + \lambda_3 \cdot y^3 + \lambda_4 \cdot y^4. \]

Therefore from equation (2.18), we have

\[ Q_4 = Q_3 + \lambda_4 \cdot [T^4 - 4 \cdot T^3 \cdot \tilde{M}_1(\hat{H}_t^p) + 6 \cdot T^2 \cdot \tilde{M}_2(\hat{H}_t^p) - 4 \cdot T \cdot \tilde{M}_3(\hat{H}_t^p) + \tilde{M}_4(\hat{H}_t^p)] \]

\[ = \lambda_0 + \lambda_1 \cdot (T - \tilde{M}_1(\hat{H}_t^p)) + \lambda_2 \cdot (T - \tilde{M}_1(\hat{H}_t^p))^2 + \lambda_3 \cdot (T - \tilde{M}_1(\hat{H}_t^p))^3 + \lambda_4 \cdot (T - \tilde{M}_1(\hat{H}_t^p))^4 \]

\[ + \lambda_2 \cdot \tilde{D}_2(\hat{H}_t^p) \]

\[ + \lambda_3 \cdot (3 \cdot T \cdot \tilde{D}_2(\hat{H}_t^p) - \tilde{D}_3(\hat{H}_t^p)) \]

\[ + \lambda_4 \cdot (6 \cdot T^2 \cdot \tilde{D}_2(\hat{H}_t^p) - 4 \cdot T \cdot \tilde{D}_3(\hat{H}_t^p) + \tilde{D}_4(\hat{H}_t^p)) \]

\[ = LFR_{t-(\mu(\hat{H}_t^p)+1)} + \lambda_2 \cdot (\sigma(\hat{H}_t^p))^2 + \lambda_3 \cdot [3 \cdot T \cdot \tilde{D}_2(\hat{H}_t^p) - \tilde{D}_3(\hat{H}_t^p)] \]

\[ + \lambda_4 \cdot [6 \cdot T^2 \cdot \tilde{D}_2(\hat{H}_t^p) - 4 \cdot T \cdot \tilde{D}_3(\hat{H}_t^p) + \tilde{D}_4(\hat{H}_t^p)] \]

(3.9)

Since

\[ 6 \cdot T^2 \cdot \tilde{D}_2(\hat{H}_t^p) - 4 \cdot T \cdot \tilde{D}_3(\hat{H}_t^p) + \tilde{D}_4(\hat{H}_t^p) = (\sigma(\hat{H}_t^p))^2 \cdot \sigma_4 \]

(3.10)

where \( \sigma_4 = 6 \cdot T^2 - 6 \cdot \mu(\hat{H}_t^p) \cdot (2 \cdot T - 1) - 4 \cdot \sigma(\hat{H}_t^p) \cdot s(\hat{H}_t^p) \cdot (T - \mu(\hat{H}_t^p)) + (\sigma(\hat{H}_t^p))^2 \cdot k(\hat{H}_t^p), \)

equation (3.9) can be rewritten as

\[ Q_4 = LFR_{t-(\mu(\hat{H}_t^p)+1)} + \lambda_2 \cdot (\sigma(\hat{H}_t^p))^2 + \lambda_3 \cdot \sigma(\hat{H}_t^p) + \lambda_4 \cdot (\sigma(\hat{H}_t^p))^2 \]

(3.11)

In this case, we have

\[ TFR_t = G_i \cdot Q_4 = G_i \left[ LFR_{t-(\mu(\hat{H}_t^p)+1)} + (\lambda_2 + \lambda_3 \cdot \sigma_4 + \lambda_4 \cdot (\sigma(\hat{H}_t^p))^2 \right] \]

(3.12)

4. Specific expressions of \( G_i \) - Assumption I

From the discussions above, we have noticed that the period quantity \( G_i = \sum_{a=15}^{49} h^p(a) \) (for year \( t \)) is very important factor in terms of linking the (period) total fertility rate to the corresponding (cohort) lifetime fertility rates.

Following a similar approach of Ryder (1964), we assume that for each age \( a \), the time sequence \( \{ h^p(a) \mid y = t-50, t-49, \ldots, t-16 \} \) can be represented by the following \( m^\text{th} \)-degree polynomial of \( y \):
\[ h_y(a) = \beta_0(a) + \beta_1(a) \cdot y + \beta_2(a) \cdot y^2 + \cdots + \beta_m(a) \cdot y^m = \sum_{i=0}^{m} [\beta_i(a) \cdot y^i] \] (4.1)

where \( m \) is a positive integer and \( \beta_i(a), \ i = 0, 1, 2, \ldots, m, \) are the polynomial coefficients. Since for each birth cohort \( y \), we have \( \sum_{a=15}^{49} h_y(a) = 1 \), it follows that

\[ \sum_{a=15}^{49} \sum_{i=0}^{m} [\beta_i(a) \cdot y^i] = \sum_{i=0}^{m} \left[ \left( \sum_{a=15}^{49} \beta_i(a) \right) \cdot y^i \right] = 1 \] (4.2)

Let \( \pi_i = \sum_{a=15}^{49} \beta_i(a) \), then we have

\[ \pi_0 + \pi_1 \cdot y + \pi_2 \cdot y^2 + \cdots + \pi_m \cdot y^m = \sum_{i=0}^{m} (\pi_i \cdot y^i) = 1 \] (4.3)

We define an \( m \)-th degree polynomial of \( y \) as follows: \( q_m(y) = \sum_{i=0}^{m} (\pi_i \cdot y^i) - 1 \), where \( \pi_m \neq 0 \), then it is obvious that the polynomial \( q_m(y) \) has 35 real (integer) roots, i.e. \( y = t-50, t-49, \ldots, t-16 \). It can be proved that when \( m < 35 \), there must be \( \pi_0 = 1 \) and \( \pi_i = 0 \) (\( i = 1, 2, \ldots, m \)). (Proof by contradiction: If \( \pi_m \neq 0 \), then according to the fundamental theorem of algebra, the polynomial \( q_m(y) \) has at most \( m \) real roots, which contradicts the fact that the polynomial \( q_m(y) \) has 35 real roots). Therefore, we have \( \pi_m = 0 \). Following the same logic, we have \( \pi_{m-1} = 0 \), \( \ldots \), \( \pi_1 = 0 \).

Hence, from equation (A.3), we have \( \pi_0 = 1 \).

From the definition of \( G_i \) and letting \( T = t - 1 \), we have

\[ G_i = \sum_{a=15}^{49} h_y^i(a) = \sum_{a=15}^{49} h_{i-15+1}(a) = \sum_{a=15}^{49} \sum_{i=0}^{m} [\beta_i(a) \cdot (T-a)^i] \] (4.4)

Then by applying the binomial theorem to equation (4.4), we have

\[ G_i = \sum_{a=15}^{49} \sum_{i=0}^{m} \left\{ \beta_i(a) \cdot \sum_{j=0}^{i} \left[ (-1)^j \cdot \frac{i!}{j!(i-j)!} \cdot T^{i-j} \cdot a^j \right] \right\} \]

\[ = \sum_{i=0}^{m} \sum_{j=0}^{i} \left[ (-1)^j \cdot \frac{i!}{j!(i-j)!} \cdot T^{i-j} \right] \cdot \sum_{a=15}^{49} [a^j \cdot \beta_i(a)] \] (4.5)

Equation (4.5) provides a general expression for \( G_i \), under assumption expressed in equation (4.1).
For each birth cohort $y$, we denote the mean age of the standardized cohort fertility schedule $h_y^c(a)$ as $\mu(\overline{h}_y^c) = \frac{1}{a - 50} \sum_{a=15}^{49} [a \cdot h_y^c(a)] = \sum_{a=15}^{49} \left[ \left( \sum_{i=0}^{a} \beta_i(a) \cdot y' \right) \cdot y' \right]$ \hspace{1cm} (4.6)

Equation (4.6) shows that under the assumption stated in equation (4.1), $\mu(\overline{h}_y^c)$ is also a polynomial of degree $m$.

For each birth cohort $y$, we denote the variance of the cohort fertility curve $h_y^c(a)$ as $\nu(\overline{h}_y^c)$, then we have

$$\nu(\overline{h}_y^c) = \sum_{a=15}^{49} [\left( h_y^c(a) - \nu(\overline{h}_y^c) \right)^2] = \sum_{a=15}^{49} \left( a^2 \cdot \left( \sum_{i=0}^{a} \beta_i(a) \cdot y' \right) \right) - (\mu(\overline{h}_y^c))^2$$

$$= \sum_{a=15}^{49} \left( \sum_{i=0}^{a} [a \cdot \beta_i(a)] \cdot y' \right) - (\mu(\overline{h}_y^c))^2$$ \hspace{1cm} (4.7)

Equation (4.7) shows that under the assumption stated in equation (4.1), $\nu(\overline{h}_y^c)$ is a polynomial of degree $2 \cdot m$.

Now, let’s consider two specific cases.

4.1 For each age $a$ $(a = 15, 16, \ldots, 49)$, the time sequence $\{h_y^c(a) \mid y = t-50, t-49, \ldots, t-16\}$ can be represented by a linear function of $y$

This is equivalent to taking $m=1$ in equation (4.1), i.e. $h_y^c(a) = \beta_0(a) + \beta_1(a) \cdot y$, where $\beta_0(a)$ is the intercept and $\beta_1(a)$ is the slope of the straight line. Then, from the discussion above, we know that $\nu_0 = \sum_{a=15}^{49} \beta_0(a) = 1$ and $\nu_1 = \sum_{a=15}^{49} \beta_1(a) = 0$. Therefore, from equation (4.5), we have

$$G_i = \sum_{a=15}^{49} \beta_0(a) + T \sum_{a=15}^{49} \beta_1(a) - \sum_{a=15}^{49} [a \cdot \beta_i(a)] = 1 - \sum_{a=15}^{49} [a \cdot \beta_i(a)]$$ \hspace{1cm} (4.8)

In the mean time, from equation (4.6), we have
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\[ \mu(h_y^c) = \sum_{a=15}^{49} [a \cdot \beta_0(a)] + \left( \sum_{a=15}^{49} [a \cdot \beta_1(a)] \right) \cdot y \]  

(4.9)

Equation (4.9) shows that under the assumption stated, the mean age of the standardized cohort fertility schedule \( h_y^c(a) \), \( y = t-50, t-49, \ldots, t-16 \), is also a linear function of the birth cohort \( y \), with the intercept being \( \sum_{a=15}^{49} [a \cdot \beta_0(a)] \) and the slope \( \sum_{a=15}^{49} [a \cdot \beta_1(a)] \). From equation (4.9), we also have

\[ \sum_{a=15}^{49} [a \cdot \beta_1(a)] = \frac{1}{34} \cdot (\mu(h_{t-16}^c) - \mu(h_{t-50}^c)) \]

Let \( \varphi = \sum_{a=15}^{49} [a \cdot \beta(a)] \), then equation (4.8) becomes \( G_i = 1 - \varphi \), where \( \varphi \) is the slope of \( \mu(h_y^c) \). In other words, under the assumption stated, \( G_i \) is equal to one minus the slope (rate of change) of the mean age of the cohort fertility curve \( h_y^c(a), y = t-50, t-49, \ldots, t-16 \).

Taking the first derivative with respect to \( y \) on both sides of equation (4.9), we obtain

\[ \left[ \mu(h_y^c) \right]_y = \sum_{a=15}^{49} [a \cdot \beta_1(a)] = \varphi \]  

(4.10)

Therefore, \( G_i \) can also be written as \( G_i = 1 - \left[ \mu(h_y^c) \right]_y \).

Under different assumptions, Ryder (1964) obtained a similar result by using the moment approach. It is obvious from equation (4.10) that (i) if the mean ages of the standardized cohort fertility curves \( h_y^c(a) \) increase from cohort to cohort (i.e. women postpone childbearing), then \( \left[ \mu(h_y^c) \right]_y > 0 \) and therefore \( G_i < 1 \); and (ii) if the mean ages of the standardized cohort fertility curves \( h_y^c(a) \) decrease from cohort to cohort (i.e. women advance childbearing), then \( \left[ \mu(h_y^c) \right]_y > 0 \) and therefore \( G_i > 1 \).

From equation (4.7), we have the variance of the cohort fertility curve \( h_y^c(a) \) \( y = t-50, t-49, \ldots, t-16 \) as follows

\[ \nu(h_y^c) = \sum_{a=15}^{49} [a^2 \cdot \beta_0(a)] + \left( \sum_{a=15}^{49} [a^2 \cdot \beta_1(a)] \right) \cdot y - (\mu(h_y^c))^2 \]  

(4.11)

Equation (4.11) shows that under the assumption stated, the variance of the cohort fertility curve
$h^c_y(a) \quad (y = t-50, t-49, \ldots, t-16)$ is a quadratic function of the birth cohort ($y$), with the coefficient of the quadratic term (i.e. $y^2$) being $-\left( \sum_{a=15}^{49} [a \cdot \beta_1(a)] \right)^2$ ($<0$). Therefore, $\nu \left( h^c_y \right) \quad (y = t-50, t-49, \ldots, t-16)$ is a parabola, which opens downwards.

$$
\mu \left( H^c_i \right) = \sum_{a=15}^{49} [a \cdot H^c_i(a)] = \frac{1}{G_i} \sum_{a=15}^{49} [a \cdot h^c_y(a)] = \frac{1}{G_i} \sum_{a=15}^{49} [a \cdot h^c_{-15}(a)]
$$

$$
= \frac{1}{G_i} \sum_{a=15}^{49} \{ a \cdot [\beta_0(a) + \beta_1(a) \cdot (t - a - 1)] \}
$$

$$
= \frac{1}{G_i} \left\{ \sum_{a=15}^{49} [a \cdot \beta_0(a)] + (t - 1) \cdot \sum_{a=15}^{49} [a \cdot \beta_1(a)] - \sum_{a=15}^{49} [a^2 \cdot \beta_1(a)] \right\} \quad (4.12)
$$

Substituting equation (4.8) into equation (4.12), we have

$$
\mu \left( H^c_i \right) = \frac{\sum_{a=15}^{49} [a \cdot \beta_0(a)] + (t - 1) \cdot \sum_{a=15}^{49} [a \cdot \beta_1(a)] - \sum_{a=15}^{49} [a^2 \cdot \beta_1(a)]}{1 - \sum_{a=15}^{49} [a \cdot \beta_1(a)]} \quad (4.13)
$$

Next, we will conduct a numerical simulation so that we can have a concrete understanding of the theoretical relationships discussed above. For this purpose, we will use the following Gamma function for the simulation.

$$
g(a) = \left\{ \begin{array}{ll}
K \cdot (a - a_0)^4 \cdot e^{-B (a - a_0)}, & \text{when} \quad a \geq a_0 \\
0, & \text{when} \quad a < a_0
\end{array} \right. \quad (4.14)
$$

The properties of the above-defined Gamma function are discussed in detail in Annex B.

Suppose that (i) the standardized cohort fertility schedule $\{ h^c_{-50}(a) \mid a = 15, 16, \ldots, 49 \}$ (i.e. the oldest birth cohort) follows a Gamma function with a mean of 28 and a standard deviation of 5, and (ii) the standardized cohort fertility schedule $\{ h^c_{-16}(a) \mid a = 15, 16, \ldots, 49 \}$ (i.e. the youngest birth cohort) follows a Gamma function with a mean of 32 and a standard deviation of 5. All the standardized cohort fertility schedules between the oldest and the youngest birth cohorts are then generated by linear interpolation age by age between the oldest and the youngest birth cohorts, i.e. for each age $a$ ($a = 15, 16, \ldots, 49$), the $h^c_y(a)$ is calculated as follows:
\[ h^c_y(a) = h^c_{y-50}(a) + \Delta(a)[y - (t - 50)], \quad y = t - 49, t - 48, \ldots, t - 17 \]  
\[ (4.15) \]

where \( \Delta(a) = [h^c_{y-16}(a) - h^c_{y-50}(a)] / 34 \). It can be easily proved that \( h^c_y(a) \) generated as per equation (4.13) satisfies \( h^c_y(a) \geq 0 \) and \( \sum_{a=15}^{49} h^c_y(a) = 1 \).

**Figure 5. Intercept by age (i.e. \( \beta_y(a) \))**

![Figure 5. Intercept by age](image1)

**Figure 6. Slope by age (i.e. \( \beta_i(a) \))**

![Figure 6. Slope by age](image2)
4.2 For each age \( a \) \((a = 15, 16, \ldots, 49)\), the time sequence \( \{ h_y^c(a) \mid y = t - 50, t - 49, \ldots, t - 16 \} \) can be represented by a quadratic function of \( y \).

This is equivalent to taking \( m = 2 \) in equation (4.1), i.e. \( h_y^c(a) = \beta_0(a) + \beta_1(a) \cdot y + \beta_2(a) \cdot y^2 \).

Then, from the discussion above, we know that \( \pi_0 = \sum_{a=15}^{49} \beta_0(a) = 1 \), \( \pi_i = \sum_{a=15}^{49} \beta_i(a) = 0 \) and \( \pi_2 = \sum_{a=15}^{49} \beta_2(a) = 0 \). Therefore, from equation (4.5), we have

\[
G_i = \sum_{a=15}^{49} \beta_0(a) - \sum_{a=15}^{49} (a \cdot \beta_1(a)) - 2 \cdot T \cdot \sum_{a=15}^{49} (a \cdot \beta_2(a)) + \sum_{a=15}^{49} (a^2 \cdot \beta_2(a)) \\
= 1 - \sum_{a=15}^{49} (a \cdot \beta_1(a)) - 2 \cdot T \cdot \sum_{a=15}^{49} (a \cdot \beta_2(a)) + \sum_{a=15}^{49} (a^2 \cdot \beta_2(a)) \quad (4.16)
\]

In this case, the mean age of the standardized cohort fertility curve \( h_y^c(a) \) is

\[
\mu(h_y^c) = \sum_{a=15}^{49} (a \cdot \beta_0(a)) + \left( \sum_{a=15}^{49} (a \cdot \beta_1(a)) \right) \cdot y + \left( \sum_{a=15}^{49} (a \cdot \beta_2(a)) \right) \cdot y^2 \quad (4.17)
\]

Equation (4.17) shows that under the assumption stated, the mean age of the cohort fertility curve \( h_y^c(a) \) is also a quadratic function of the birth cohort \( y \). The variance of \( h_y^c(a) \) is

\[
v(h_y^c) = \sum_{a=15}^{49} \left[ (a - \mu(h_y^c))^2 \cdot h_y^c(a) \right] = \sum_{a=15}^{49} (a^2 \cdot h_y^c(a)) - (\mu(h_y^c))^2
\]
\[
\sum_{a=15}^{49} [a \cdot \beta_0(a)] + \left( \sum_{a=15}^{49} [a \cdot \beta_1(a)] \right) \cdot y + \left( \sum_{a=15}^{49} [a \cdot \beta_2(a)] \right) \cdot y^2 - (\mu(h^c_\nu))^2 \quad (4.18)
\]

Equation (4.18) shows that under the assumption stated, the variance of the cohort fertility curve \( h^c_\nu(a) \) is a quartic function of the birth cohort \((y)\). From equation (4.16), we obtain

\[
(\mu(h^c_\nu))^2 + v(h^c_\nu) = \sum_{a=15}^{49} [a \cdot \beta_0(a)] + \left( \sum_{a=15}^{49} [a \cdot \beta_1(a)] \right) \cdot y + \left( \sum_{a=15}^{49} [a \cdot \beta_2(a)] \right) \cdot y^2 \quad (4.19)
\]

Taking the second derivative with respect to \(y\) on both sides of equations (4.17) and (4.19), we have

\[
[\mu(h^c_\nu)]'' = 2 \cdot \sum_{a=15}^{49} [a \cdot \beta_2(a)] \quad (4.20)
\]

\[
[(\mu(h^c_\nu))^2 + v(h^c_\nu)]'' = 2 \cdot \sum_{a=15}^{49} [a \cdot \beta_2(a)] \quad (4.21)
\]

Incorporating equations (4.20) and (4.21) into equation (4.14), we get

\[
G_t = 1 - \sum_{a=15}^{49} [a \cdot \beta_1(a)] - T \cdot [\mu(h^c_\nu)]'' + \frac{1}{2} \cdot [(\mu(h^c_\nu))^2]'' + \frac{1}{2} \cdot [v(h^c_\nu)]'' \quad (4.22)
\]

From equation (4.17), we have

\[
\mu(h^c_{t-50}) = \sum_{a=15}^{49} [a \cdot \beta_0(a)] + \left( \sum_{a=15}^{49} [a \cdot \beta_1(a)] \right) \cdot (T - 49) + \left( \sum_{a=15}^{49} [a \cdot \beta_2(a)] \right) \cdot (T - 49)^2 \quad (4.23)
\]

\[
\mu(h^c_{t-16}) = \sum_{a=15}^{49} [a \cdot \beta_0(a)] + \left( \sum_{a=15}^{49} [a \cdot \beta_1(a)] \right) \cdot (T - 15) + \left( \sum_{a=15}^{49} [a \cdot \beta_2(a)] \right) \cdot (T - 15)^2 \quad (4.24)
\]

where \( T = t - 1 \). Subtracting equation (4.23) from equation (4.24), we obtain

\[
\mu(h^c_{t-16}) - \mu(h^c_{t-50}) = 34 \cdot \sum_{a=15}^{49} [a \cdot \beta_1(a)] + (68 \cdot T - 2176) \cdot \sum_{a=15}^{49} [a \cdot \beta_2(a)]
\]

\[
= 34 \cdot \sum_{a=15}^{49} [a \cdot \beta_1(a)] + \left( \frac{68 \cdot T - 2176}{2} \right) \cdot [\mu(h^c_\nu)]''
\]

\[
= 34 \cdot \sum_{a=15}^{49} [a \cdot \beta_1(a)] + (34 \cdot T - 1088) \cdot [\mu(h^c_\nu)]'' \quad (4.25)
\]

Therefore, we have

\[
\sum_{a=15}^{49} [a \cdot \beta_1(a)] = \frac{1}{34} \left[ (\mu(h^c_{t-16}) - \mu(h^c_{t-50})) - (34 \cdot T - 1088) \cdot [\mu(h^c_\nu)]'' \right] \quad (4.26)
\]

Finally, equation (4.22) becomes
\[ G_t = 1 - \frac{1}{34} \left( (\mu(h_{t-16}^c) - \mu(h_{t-50}^c)) - (34 \cdot T - 1088) \cdot [\mu(h_{t}^c)]_y^* \right) \]
\[ - T \cdot [\mu(h_{t}^c)]_y^* + \frac{1}{2} \cdot [(\mu(h_{t}^c))^2]_y^* + \frac{1}{2} \cdot [v(h_{t}^c)]_y^* \]
\[ = 1 - \frac{1}{34} \left( (\mu(h_{t-16}^c) - \mu(h_{t-50}^c)) - (35 \cdot T - 1088) \cdot [\mu(h_{t}^c)]_y^* \right) \]
\[ + \frac{1}{2} \cdot [(\mu(h_{t}^c))^2]_y^* + \frac{1}{2} \cdot [v(h_{t}^c)]_y^* \]  

(4.27)

where \( \mu(h_{t-16}^c) - \mu(h_{t-50}^c) \) can be regarded as the amount of “shift” between the two standardized cohort fertility schedules \( h_{t-16}^c(a) \) and \( h_{t-50}^c(a) \).

5. A specific expression of \( G_t \) - Assumption II

We assume that, for each birth cohort \( y \) (\( y = t-50, t-49, \ldots, t-16 \)), its standardized fertility schedule \( h_t^c(a) \) is a continuous function of age \( a \). Therefore, we have \( h_t^c(a) \geq 0 \) and \( \int_{15}^{50} h_t^c(a) da = 1 \). In addition, we designate the birth cohort \( t-50 \) (i.e. the birth cohort that reached the oldest childbearing age at the beginning of year \( t \)) as the benchmark cohort. For \( h_t^c(a) \), we symbolize its mean, variance, skewness and kurtosis as follows:

Mean:
\[ \mu(h_t^c) = \int_{15}^{50} [a \cdot h_t^c(a)] da \]  

(5.1)

Variance:
\[ v(h_t^c) = \int_{15}^{50} [(a - \mu(h_t^c))^2 \cdot h_t^c(a)] da \]  

(5.2)

Skewness:
\[ s(h_t^c) = \int_{15}^{50} \left[ \frac{a - \mu(h_t^c)}{\sigma(h_t^c)} \right]^3 \cdot h_t^c(a) \]  

(5.3)

Kurtosis:
\[ k(h_t^c) = \int_{15}^{50} \left[ \frac{a - \mu(h_t^c)}{\sigma(h_t^c)} \right]^4 \cdot h_t^c(a) \]  

(5.4)

where \( \sigma(h_t^c) = \sqrt{v(h_t^c)} \) represents the standard deviation of \( h_t^c(a) \).
Now, we assume that each cohort curve \( h^c_y(a) \), \( y = t-49, t-48, \ldots, t-16 \), shifts along the age-axis by a constant amount \( \delta \) (with no change in the shape of the curve, see Figure 8) relative to the curve of the preceding birth cohort (i.e. \( y-1 \)), i.e. \( h^c_y(a) = h^c_{y-1}(a - \delta) \), \( y = t-49, t-48, \ldots, t-16 \). Hence, we have \( h^c_{t-(a+1)}(a) = h^c_{t-50}(a - (49 - a) \cdot \delta) \). Therefore, from equation (5.1), we have

\[
\mu(h^c_y) = \int_{15}^{50} [a \cdot h^c_y(a)] da = \int_{15}^{50} [a \cdot h^c_{y-1}(a - \delta)] da \tag{5.5}
\]

Let \( u = a - \delta \), then \( a = u + \delta \) and \( da = du \). Therefore, from equation (5.5), we obtain

\[
\mu(h^c_y) = \int_{15}^{50} [(u + \delta) \cdot h^c_{y-1}(u)] du = \int_{15}^{50} [u \cdot h^c_{y-1}(u)] du + \delta \int_{15}^{50} h^c_{y-1}(u) du = \mu(h^c_{y-1}) + \delta \tag{5.6}
\]

Equation (5.6) shows that under the assumption stated, the mean age of \( h^c_y(a) \) also shifts towards the same direction (when \( \delta > 0 \), the standardized cohort fertility curves shift towards the right side of the age-axis (i.e. higher ages); when \( \delta < 0 \), the standardized cohort fertility curves shift towards the left side of the age-axis (i.e. lower ages)) and by the same amount (i.e. \( |\delta| \)) as compared to \( h^c_{y-1}(a) \). From equation (5.6), we have

\[
\mu(h^c_y) = \mu(h^c_{t-50}) + [y - (t - 50)] \cdot \delta, \quad y = t-50, t-49, \ldots, t-16 \tag{5.7}
\]

Therefore, \( \mu(h^c_{y-1}) \) is a linear function of \( y \). Taking the first derivative with respect to \( y \) on both sides, we obtain \( [\mu(h^c_y)]'_y = \delta \).

Similarly, it can be proved that \( v(h^c_y) = v(h^c_{y-1}) \), \( s(h^c_y) = s(h^c_{y-1}) \), and \( k(h^c_y) = k(h^c_{y-1}) \).

From the discussions above, we have

\[
G_t = \sum_{a=15}^{49} h^c_y(a) = \sum_{a=15}^{49} h^c_{t-(a+1)}(a) = \sum_{a=15}^{49} h^c_{t-50}(a - (49 - a) \cdot \delta) = \sum_{a=15}^{49} h^c_{t-50}((1 + \delta) \cdot a - 49 \cdot \delta) \tag{5.8}
\]

In continuous form, equation (5.8) can be written as

\[
G_t = \int_{15}^{50} h^c_{t-50}((1 + \delta) \cdot a - 49 \cdot \delta) da \tag{5.9}
\]

Let \( u = (1 + \delta) \cdot a - 49 \cdot \delta \), then we have \( da = du/(1 + \delta) \), where \( \delta \neq -1 \). Therefore, equation (5.9) becomes
Equation (5.10) shows that $G_i$ is a decreasing function of $\delta$ (i.e. the larger the $\delta$, the smaller the $G_i$). Equation (5.9) also shows that (i) when all the concerned standardized cohort fertility curves are exactly the same (i.e. $\delta = 0$), we have $G_i = 1$; (ii) when the concerned standardized cohort fertility curves shift to the right (i.e. to higher ages, but with no change in the shape) by the same amount (i.e. $\delta > 0$) from one cohort to the next (i.e. women postpone childbearing), we have $G_i < 1$; (iii). when the concerned standardized cohort fertility curves shift to the left (i.e. to lower ages, but with no change in the shape) by the same amount (i.e. $\delta < 0$) from one cohort to the next (i.e. women advance childbearing), we have $G_i > 1$. For example, if $\delta = 0.1$, then $G_i = 0.91$; if $\delta = -0.1$, then $G_i = 1.11$.

Equation (5.10) also shows that $G_i$ is a non-linear function of $\delta$ (i.e. a hyperbola with $\delta \neq -1$). But when $-0.10 \leq \delta \leq 0.10$, $G_i$ is very close to a linear function of $\delta$ (with the coefficient of determination $R^2 = 0.997$), i.e. on the interval $[-0.10, 0.10]$, we have $G_i \approx 1 - \delta$. (Actually, the
power series expansion can be considered: \( \frac{1}{1+\delta} = 1 - \delta + \delta^2 - \delta^3 + \delta^4 - \cdots \), where \(|\delta| < 1\). Since \( [\mu(h^c_y)]_t = \delta \), we have \( G_t = 1 - [\mu(h^c_y)]_t \). Here, we notice that \( G_t \) is determined by the rate of change \((\mu(h^c_y))_t\) in the mean age of the standardized cohort fertility schedule \((h^c_y(a))\) almost in the same form, under the two different assumptions (i.e. I and II).

Under a stricter assumption (i.e. constant cohort and period quanta), Zeng and Land (2002) obtained a similar result as the one expressed in equation (5.9), who used the symbol \( r_c \) in their paper. However, our analysis above shows that the assumption of constant cohort and period quanta is not necessary for the result to hold.

Now, let’s take a look at the relationship between the mean ages of the standardized period and cohort fertility curves. From the discussions above, we have

\[
\mu(H^p_y) = \frac{1}{G_t} \sum_{a=15}^{49} [a \cdot H^p_y(a)] = \frac{1}{G_t} \sum_{a=15}^{49} [a \cdot h^c_y(a)] = \frac{1}{G_t} \sum_{a=15}^{49} [a \cdot h^c_{y-(a+1)}(a)]
\]

\[
= \frac{1}{G_t} \sum_{a=15}^{49} [a \cdot h^c_{y-50}((1+\delta) \cdot a - 49 \cdot \delta)]
\]

(5.11)

In continuous form, equation (5.11) can be written as

\[
\mu(H^p_y) = \frac{1}{G_t} \int_{15}^{50} [a \cdot h^c_{y-50}((1+\delta) \cdot a - 49 \cdot \delta)] da
\]

(5.12)

Let \( u = (1+\delta) \cdot a - 49 \cdot \delta \), then we have \( a = (u + 49 \cdot \delta) / (1+\delta) \) and \( da = du / (1+\delta) \), where \( \delta \neq -1 \). Therefore, equation (5.12) can be written as

\[
\mu(H^p_y) = \frac{1}{G_t \cdot (1+\delta)^2} \int_{15}^{50} [(u + 49 \cdot \delta) \cdot h^c_{y-50}(u)] du
\]

(5.13)

Since \( G_t = 1 / (1+\delta) \), equation (5.13) becomes

\[
\mu(H^p_y) = \frac{1}{1+\delta} \int_{15}^{50} [(u + 49 \cdot \delta) \cdot h^c_{y-50}(u)] du = \frac{1}{1+\delta} \left[ \int_{15}^{50} [u \cdot h^c_{y-50}(u)] du + 49 \cdot \delta \int_{15}^{50} h^c_{y-50}(u) du \right]
\]

\[
= \frac{1}{1+\delta} \left( \mu(h^c_{y-50}) + 49 \cdot \delta \right) = \mu(h^c_{y-50}) + \frac{\delta}{1+\delta} \cdot (49 - \mu(h^c_{y-50}))
\]

(5.14)

where \( \mu(h^c_{y-50}) = \int_{15}^{50} [u \cdot h^c_{y-50}(u)] du \) is the mean age of the standardized fertility schedule of the
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benchmark birth cohort (i.e. birth cohort $t-50$). Taking the first derivative with respect to $\delta$ on both sides of equation (5.14), we obtain $[\mu(H_p^t)]_\delta = (49 - \mu(h_{t-50}^c)) \cdot \frac{1}{(1+\delta)^2} > 0$. Therefore, $\mu(H_p^t)$ is an increasing function of $\delta$ (i.e. the larger the $\delta$, the lager the $\mu(H_p^t)$).

Since $\mu(h_{t-16}^c) = \mu(h_{t-50}^c) + 34 \cdot \delta$, it follows from equation (5.14)

$$\mu(H_p^t) = \frac{1}{1+\delta} \cdot (\mu(h_{t-16}^c) + 15 \cdot \delta) = \mu(h_{t-16}^c) - \delta \cdot (\mu(h_{t-16}^c) - 15)$$

(5.15)

where $\mu(h_{t-16}^c) = \int_{15}^{50} [u \cdot h_{t-16}^c(u)] du$ is the mean age of the standardized age-specific fertility curve of the birth cohort $t-16$.

From equations (5.14) and (5.15), it is obvious that (i) if $\delta = 0$, then $\mu(H_p^t) = \mu(h_{t-50}^c)$, (ii) if $\delta > 0$, then $\mu(h_{t-50}^c) < \mu(H_p^t) < \mu(h_{t-16}^c)$, (iii) if $\delta < 0$, then $\mu(h_{t-16}^c) < \mu(H_p^t) < \mu(h_{t-50}^c)$. For example, assume $\mu(h_{t-50}^c) = 28$, then $\mu(h_{t-16}^c) = 28 + 34 \cdot \delta$. Therefore, if $\delta = 0.1$, then $\mu(h_{t-16}^c) = 31.40$ and $\mu(H_p^t) = 29.91$; if $\delta = -0.1$, then $\mu(h_{t-16}^c) = 24.60$ and $\mu(H_p^t) = 25.67$.

Next, let’s take a look at the relationship between the variances of the standardized period and cohort fertility curves. From the discussions above, we have

$$v(H_p^t) = \sum_{a=15}^{49} [(a - H(h_{t-50}^c))^2 \cdot H_p^t(a)] = \frac{1}{G_t} \cdot \sum_{a=15}^{49} [(a - \mu(H_p^t))^2 \cdot h_p^t(a)]$$

$$= \frac{1}{G_t} \cdot \sum_{a=15}^{49} [(a - \mu(H_p^t))^2 \cdot h_{t-16}(a)]$$

$$= \frac{1}{G_t} \cdot \sum_{a=15}^{49} [(a - \mu(H_p^t))^2 \cdot h_{t-50}^c((1+\delta) \cdot a - 49 \cdot \delta)]$$

(5.16)

In continuous form, equation (5.16) can be written as

$$v(H_p^t) = \frac{1}{G_t} \cdot \int_{15}^{50} [(a - \mu(H_p^t))^2 \cdot h_{t-50}^c((1+\delta) \cdot a - 49 \cdot \delta)] da$$

(5.16)

Let $u = (1+\delta) \cdot a - 49 \cdot \delta$, then we have $a = (u + 49 \cdot \delta)/(1+\delta)$ and $da = du/(1+\delta)$, where $\delta \neq -1$. Therefore, equation (5.16) can be written as
\[
v(H^p_t) = \frac{1}{G_t \cdot (1 + \delta)} \cdot \int_{15}^{50} \left( \frac{u + 49 \cdot \delta}{1 + \delta} - \mu(H^p_t) \right)^2 \cdot h^c_{t-50}(u) \, du \\
= \frac{1}{G_t \cdot (1 + \delta)} \cdot \int_{15}^{50} \left( \frac{u + 49 \cdot \delta}{1 + \delta} + 49 \cdot \delta \right)^2 \cdot h^c_{t-50}(u) \, du \\
= \frac{1}{G_t \cdot (1 + \delta)} \cdot \int_{15}^{50} \left( u - \mu(h^c_{t-50}) \right)^2 \cdot h^c_{t-50}(u) \, du \\
\] (5.18)

Since \( G_t = 1/(1 + \delta) \), equation (5.18) becomes

\[
v(H^p_t) = \frac{1}{(1 + \delta)^2} \cdot \int_{15}^{50} \left[ (u - \mu(h^c_{t-50}))^2 \cdot h^c_{t-50}(u) \right] du = \frac{1}{(1 + \delta)^2} \cdot v(h^c_{t-50}) = \left( \frac{\sigma(h^c_{t-50})}{1 + \delta} \right)^2 \] (5.18)

Equivalently, we have \( \sigma(H^p_t) = \sigma(h^c_{t-50})/(1 + \delta) \), where \( \sigma \) stands for standard deviation. Equation (5.18) shows that \( v(H^p_t) \) is a decreasing function of \( \delta \) (i.e. the larger the \( \delta \), the smaller the \( v(H^p_t) \)). It is obvious that (i) if \( \delta = 0 \), then \( \sigma(H^p_t) = \sigma(h^c_{t-50}) \); (ii) if \( \delta > 0 \), then \( \sigma(H^p_t) < \sigma(h^c_{t-50}) \); (iii) if \( \delta < 0 \), then \( \sigma(H^p_t) > \sigma(h^c_{t-50}) \). For example, assume \( \delta = 4 \), then if \( \delta = 0.1 \), then \( \sigma(H^p_t) = 3.64 \); if \( \delta = -0.1 \), then \( \sigma(H^p_t) = 4.44 \).

Similarly, it can be proved that

\[
s(H^p_t) = \int_{15}^{50} \left( \frac{a - \mu(H^p_t)}{\sigma(H^p_t)} \right)^3 \cdot H^p_t(a) \, da = s(h^c_{t-50}) \] (5.20)

\[
k(H^p_t) = \int_{15}^{50} \left( \frac{a - \mu(H^p_t)}{\sigma(H^p_t)} \right)^4 \cdot H^p_t(a) \, da = k(h^c_{t-50}) \] (5.21)

where \( s \) and \( k \) stand for skewness and kurtosis respectively.

Now, we discuss the relationship between the period and the cohort mean ages. It is obvious that birth cohort \( y = t - (\mu(H^p_t) + 1) \) is aged \( \mu(H^p_t) \) (when \( \mu(H^p_t) \) is an integer) at the beginning of year \( t \). From equations (5.7) and (5.14), we have
The above discussion indicates that, under the assumption stated, the mean age of the standardized period fertility schedule $H_p(t)$ is the same as the mean age of the standardized fertility schedule of the birth cohort $y = t - (\mu(H_p) + 1)$ that reaches its mean age of fertility right at the beginning of year $t$.

### 6. Relationship between the two period curves $f_p(t)$ and $h_p(t)$

In the discussions above, there are two important period fertility curves for year $t$, i.e. $f_p(t)$ and $h_p(t)$, whose general relationship is given in equation (2.4). Now, we will look at the relationships between the positions and shapes of the two period curves.

Let $F_p(t) = f_p(t)/TFR_y$, $H_p(t) = h_p(t)/G_y$, then we have (i) $F_p(t) \geq 0$, $\sum_{a=15}^{49} F_p(t) = 1$, and (ii) $H_p(t) \geq 0$, $\sum_{a=15}^{49} H_p(t) = 1$. Hence, equation (2.4) can be rewritten as

$$F_p(t) = \frac{G_y}{TFR_y} \cdot H_p(t) \cdot LFR_{t-(a+1)} = \frac{G_y}{TFR_y} \cdot H_p(t) = \sum_{i=0}^{n} (\lambda_i \cdot y^i)$$

where $T = t - 1$. Suppose that $LFR_y$ can be expressed by the following $n^{th}$-degree polynomial of $y$:

$$LFR_y = \lambda_0 + \lambda_1 \cdot y + \lambda_2 \cdot y^2 + \cdots + \lambda_n \cdot y^n = \sum_{i=0}^{n} (\lambda_i \cdot y^i)$$

where $n$ is a non-negative integer and $\lambda_i$, $(i = 0, 1, 2, \ldots, n)$, are the polynomial coefficients. Then equation (6.1) becomes

$$F_p(t) = \frac{G_y}{TFR_y} \cdot H_p(t) \cdot \sum_{i=0}^{n} [\lambda_i \cdot (T-a)^i]$$

Applying the binomial theorem on $(T-a)^i$, we have
\[ F_t^p(a) = \frac{G}{TFR_t} \cdot H_t^p(a) \cdot \sum_{i=0}^{n} \left[ \lambda_t \cdot \sum_{j=0}^{i} \left[ (-1)^j \cdot \left( \frac{t^j}{j!(i-j)!} \right) \cdot T^{i-j} \cdot a^j \right] \right] \]

\[ = \frac{G}{TFR_t} \cdot \sum_{i=0}^{n} \left[ \lambda_t \cdot \sum_{j=0}^{i} \left[ (-1)^j \cdot \left( \frac{t^j}{j!(i-j)!} \right) \cdot T^{i-j} \cdot a^j \cdot H_t^p(a) \right] \right] \]  
\quad (6.4)

Define the \( r \)th absolute moment (about zero or origin) of \( F_t^p(a) \) and \( H_t^p(a) \) as follows:

\[ \hat{M}_r\{F_t^p\} = \sum_{a=15}^{49} [a^r \cdot F_t^p(a)], \quad r = 0, 1, 2, \ldots \]  
\quad (6.5)

\[ \hat{M}_r\{H_t^p\} = \sum_{a=15}^{49} [a^r \cdot H_t^p(a)], \quad r = 0, 1, 2, \ldots \]  
\quad (6.6)

Then, from equation (6.4), we have

\[ \hat{M}_r\{F_t^p\} = \frac{G}{TFR_t} \cdot \sum_{i=0}^{n} \left[ \lambda_t \cdot \sum_{j=0}^{i} \left[ (-1)^j \cdot \left( \frac{t^j}{j!(i-j)!} \right) \cdot T^{i-j} \cdot \sum_{a=15}^{49} [a^{j+r} \cdot H_t^p(a)] \right] \right] \]

\[ = \frac{G}{TFR_t} \cdot \sum_{i=0}^{n} \left[ \lambda_t \cdot \sum_{j=0}^{i} \left[ (-1)^j \cdot \left( \frac{t^j}{j!(i-j)!} \right) \cdot T^{i-j} \cdot \hat{M}_{j+r}\{H_t^p\} \right] \right] \quad (6.7) \]

Equation (6.7) provides a general expression for the relationship between the absolute moments of \( F_t^p(a) \) and \( H_t^p(a) \).

Now, let’s consider one specific case, where \( LFR_y \) is a linear function of \( y \). In this case, we have.

\( LFR_y = \lambda_0 + \lambda \cdot y \). Therefore, equation (6.7) simplifies to

\[ \hat{M}_r\{F_t^p\} = \frac{G}{TFR_t} \cdot \left\{ (\lambda_0 + \lambda_t \cdot T) \cdot \sum_{a=15}^{49} [a^r \cdot H_t^p(a)] - \lambda_t \sum_{a=15}^{49} [a^{r+1} \cdot H_t^p(a)] \right\} \]

\[ = \frac{G}{TFR_t} \cdot [(\lambda_0 + \lambda \cdot T) \cdot \hat{M}_r\{H_t^p\} - \hat{M}_{r+1}\{H_t^p\}] \quad (6.8) \]

Specially, the mean age of \( F_t^p(a) \) is as follows:

\[ \mu\{F_t^p\} = \sum_{a=15}^{49} [a \cdot F_t^p(a)] = \frac{G}{TFR_t} \cdot [(\lambda_0 + \lambda_t \cdot T) \cdot \mu\{H_t^p\} - \mu\{H_t^p\} + \mu\{H_t^p\}^2] \]

\[ = \frac{G}{TFR_t} \cdot [(\lambda_0 + \lambda_t \cdot T) \cdot (\mu\{H_t^p\} - \mu\{H_t^p\}^2)] \]

\[ = \frac{G}{TFR_t} \cdot [(\lambda_0 \cdot \mu\{H_t^p\} + \lambda_t \cdot (T \cdot \mu\{H_t^p\} - (\mu\{H_t^p\})^2) - \mu\{H_t^p\}]) \quad (6.9) \]
Since \(LFR_y = \lambda_0 + \lambda_i \cdot y\), we have \(LFR_y - LFR_{y-1} = \lambda_i\) \((y = t - 49, t - 48, \ldots, t - 16)\), and therefore,
\[
LFR_y = LFR_{t-50} + \lambda_i[y - (t - 50)]
\]  (6.10)

Based on equation (3.3), equation (6.1) can be rewritten as
\[
F_t^p(a) = \frac{G_{t}^{LFR}}{TFR_t} \cdot H_t^p(a) \cdot LFR_{t-(a+1)} = \left(\frac{LFR_{t-(a+1)}}{LFR_{t-(\mu(H_p^{t+1})}+1}\right) \cdot H_t^p(a)
\]  (6.11)

It is obvious from equation (6.11) that \(F_t^p(\mu(H_p^{t+1})) = H_t^p(\mu(H_p^{t+1}))\). Further
\[
F_t^p(a) - H_t^p(a) = \left(\frac{LFR_{t-(a+1)} - LFR_{t-(\mu(H_p^{t+1})+1)}}{LFR_{t-(\mu(H_p^{t+1})+1)}\right) \cdot H_t^p(a)
\]  (6.12)

From equations (6.10) and (6.12), we obtain
\[
F_t^p(a) - H_t^p(a) = \lambda_i \left(\frac{\mu(H_p^{t+1}) - a}{LFR_{t-(\mu(H_p^{t+1})+1)}\right) \cdot H_t^p(a) = \frac{\lambda_i}{LFR_{t-(\mu(H_p^{t+1})+1)}\right) \cdot (\mu(H_p^{t+1}) - a) \cdot H_t^p(a)
\]  (6.13)

From equation (6.13), we obtain
\[
\sum_{a=15}^{49} [a \cdot F_t^p(a)] - \sum_{a=15}^{49} [a \cdot H_t^p(a)] = \frac{\lambda_i}{LFR_{t-(\mu(H_p^{t+1})+1)}\right) \cdot \sum_{a=15}^{49} [(\mu(H_p^{t+1}) \cdot a - a^2) \cdot H_t^p(a)]
\]  (6.14)

that is
\[
\mu(F_t^p) - \mu(H_t^p) = \frac{\lambda_i}{LFR_{t-(\mu(H_p^{t+1})+1)}\right) \cdot \sum_{a=15}^{49} [(\mu(H_p^{t+1}) \cdot a - a^2) \cdot H_t^p(a)]
\]

\[
= \frac{\lambda_i}{LFR_{t-(\mu(H_p^{t+1})+1)}\right) \cdot \mu(H_p^{t+1}) \cdot \sum_{a=15}^{49} [(a \cdot H_t^p(a)] - \sum_{a=15}^{49} a^2 \cdot H_t^p(a)]
\]

\[
= \frac{\lambda_i}{LFR_{t-(\mu(H_p^{t+1})+1)}\right) \cdot (\mu(H_p^{t+1})^2 - \sum_{a=15}^{49} a^2 \cdot H_t^p(a)]
\]  (6.14)

\[
= -\lambda_i \cdot \left(\frac{\psi(H_p^{t+1})}{LFR_{t-(\mu(H_p^{t+1})+1)}\right)
\]  (6.14)

Therefore, when \(\lambda_i > 0\), we have \(\mu(F_t^p) < \mu(H_t^p)\), when \(\lambda_i < 0\), we have \(\mu(F_t^p) > \mu(H_t^p)\).
Based on equation (6.10), we can rewrite equation (6.14) as

$$\mu(F^p_t) - \mu(H^p_t) = -\frac{\lambda_i \cdot v(H^p_t)}{LFR_{t-50} + \lambda_i \cdot (49 - \mu(H^p_t))} \quad (6.15)$$

Equation (6.15) also shows that $\mu(F^p_t) - \mu(H^p_t)$ is a hyperbolic function of $\lambda_i$. Taking the first derivative with respect to $\lambda_i$ on both sides of equation (6.15), we obtain

$$[\mu(F^p_t) - \mu(H^p_t)] \lambda_i = -\frac{v(H^p_t) \cdot LFR_{t-50}}{[LFR_{t-50} + \lambda_i \cdot (49 - \mu(H^p_t))]^2} < 0 \quad (6.16)$$

Therefore, $\mu(F^p_t) - \mu(H^p_t)$ is a decreasing function of $\lambda_i$. In other words, $|\mu(F^p_t) - \mu(H^p_t)|$ is an increasing function of $|\lambda_i|$. Figure 9 graphs the relationship between $\mu(F^p_t) - \mu(H^p_t)$ and $\lambda_i$ (assuming $LFR_{t-50} = 5$, $\mu(H^p_t) = 30$, and $v(H^p_t) = 25$).

**Figure 9. Relationship between $\mu(F^p_t) - \mu(H^p_t)$ and $\lambda_i$.**

![Graph showing the relationship between $\mu(F^p_t) - \mu(H^p_t)$ and $\lambda_i$.]

7. A closer examination of the Ryder’s basic translation equation

In his classic paper on demographic translation, Ryder (1964) developed the following basic translation equation between period total fertility rate and cohort total fertility rate:

$$B(0,T + \mu_i) = [\beta(0,T)] \cdot [1 - \mu(T)] \quad (7.1)$$

In normal term, equation (7.1) is equivalent to

$$TFR_{T+\mu_i} = LFR_T \cdot [1 - \mu(T)] \quad (7.2)$$
Ryder arrived at the relationship in equation (7.1) based on the assumption that for each age, the time series of the age-specific fertility rates may be represented by an $n^{th}$-degree polynomial with respect to $T$, where $T$ denotes the birth cohort (i.e. year of birth).

Following the approach of Ryder (1964), we assume that for each age $a \in (15, 16, \ldots, 49)$, the time series $\{f_y^c(a) | y = t-50, t-49, \ldots, t-16\}$ can be represented by the following $n^{th}$-degree polynomial of $y$:

$$f_y^c(a) = \rho_0(a) + \rho_1(a) \cdot y + \rho_2(a) \cdot y^2 + \cdots + \rho_n(a) \cdot y^n = \sum_{i=0}^{n} [\rho_i(a) \cdot y^i]$$  \hspace{1cm} (7.3)

where $n$ is a non-negative integer and $\rho_i(a), i = 0, 1, 2, \ldots, n,$ are the polynomial coefficients.

Then, from equation (2.2), we obtain the cohort total fertility rate:

$$LFR_y = \sum_{a=15}^{49} f_y^c(a) = \sum_{a=15}^{49} \left\{ \sum_{i=0}^{n} [\rho_i(a) \cdot y^i] \right\} = \sum_{i=0}^{n} \left[ \sum_{a=15}^{49} \rho_i(a) \right] \cdot y^i$$  \hspace{1cm} (7.4)

It is obvious from equation (7.4) that under the assumption expressed in equation (7.3), the cohort total fertility rate (i.e. $LFR_y$) is also an $n^{th}$-degree polynomial with respect to $y$.

Similarly, from equation (2.5), we obtain the period total fertility rate:

$$TFR_y = \sum_{a=15}^{49} f_{t-a+1}^c(a) = \sum_{a=15}^{49} \left\{ \sum_{i=0}^{n} [\rho_i(a) \cdot (t-(a+1))^i] \right\} = \sum_{i=0}^{n} \left( \sum_{a=15}^{49} \rho_i(a) \cdot (T-a)^i \right)$$  \hspace{1cm} (7.5)

where $T = t-1$. Using the binomial theorem, we have

$$TFR_y = \sum_{a=15}^{49} \left( \sum_{i=0}^{n} \left( \sum_{j=0}^{i} \left( -1 \right)^j \cdot \frac{i!}{j!(i-j)!} \cdot T^{i-j} \cdot a^j \cdot \rho_i(a) \right) \right)$$  \hspace{1cm} (7.6)

It is obvious from equation (7.6) that under the assumption expressed in equation (7.3), the period total fertility rate (i.e. $TFR_y$) is an $n^{th}$-degree polynomial with respect to $t$.

Under the assumption expressed in equation (7.3), we have the mean age of childbearing of birth cohort $y$:

$$\mu \left( f_y^c \right) = \sum_{a=15}^{49} \left[ a \cdot \left( \frac{f_y^c(a)}{LFR_y} \right) \right] = \sum_{a=15}^{49} \left[ a \cdot h_y^c(a) \right]$$  \hspace{1cm} (7.7)
where $h_y^i(a) = f_y^i(a) / LFR_y$. In this case, it is obvious that $h_y^i(a)$ is a ratio of two $n$th-degree polynomials with respect to $y$. Therefore, even the first derivative of $\mu(f_y^i)$ is a very complex function of $y$.

Now, we look at a very special situation. Let’s assume that $LFR_y$ is a constant (denoted as $LFR$), then equation (7.4) becomes

$$\sum_{i=0}^{n} \left[ \sum_{a=15}^{49} \rho_i(a) \right] \cdot y^i = LFR$$  

Taking the $n$th derivative on both sides of equation (7.8), we have $n! \sum_{a=15}^{49} \rho_n(a) = 0$. Therefore,

$$\sum_{a=15}^{49} \rho_n(a) = 0.$$  

Similarly, it can be proved that $\sum_{a=15}^{49} \rho_{n-1}(a) = 0$, …, $\sum_{a=15}^{49} \rho_1(a) = 0$. And finally,

$$\sum_{a=15}^{49} \rho_0(a) = LFR.$$  

In this case, equation (7.7) becomes

$$\mu(f_y^i) = \frac{1}{LFR} \cdot \sum_{a=15}^{49} \left[ a \cdot f_y^i(a) \right] = \frac{1}{LFR} \cdot \sum_{a=15}^{49} \left\{ a \cdot \sum_{i=0}^{n} \rho_i(a) \cdot y^i \right\}$$

$$= \frac{1}{LFR} \cdot \sum_{i=0}^{n} \left\{ \sum_{a=15}^{49} \left[ a \cdot \rho_i(a) \right] \cdot y^i \right\}$$  

Equation (7.9) shows that under the assumptions stated, $\mu(f_y^i)$ is an $n$th-degree polynomial with respect to $y$. It follows that the first derivative of $\mu(f_y^i)$ with respect to $y$ is an $(n-1)$th polynomial of $y$. Therefore, the first derivative of $\mu(f_y^i)$ is constant if and only if $n = 1$.

Under the assumptions that (i) for each age $a$ (where $a = 15, 16, \ldots, 49$), the time series $f_y^i(a)$, $y = t-50, t-49, \ldots, t-16$, can be represented by a linear function of $y$, and (ii) $LFR_y$ is constant with respect to $y$ (denoted as $LFR$), we have from equations (7.4) and (7.5)

$$LFR = \sum_{a=15}^{49} \rho_0(a)$$  

$$TFR = \sum_{a=15}^{49} \rho_0(a) + \sum_{a=15}^{49} \left[ \rho_i(a) \cdot (T - a) \right] = \sum_{a=15}^{49} \rho_0(a) - \sum_{a=15}^{49} \left[ a \cdot \rho_i(a) \right]$$
Combining equations (7.10) and (7.11), we obtain

\[ TFR_t = LFR - \sum_{a=15}^{49} [a \cdot \rho_t(a)] \]  

(7.12)

From equation (7.9), we have

\[ \mu(f_y^c) = \frac{1}{LFR} \left[ \sum_{a=15}^{49} [a \cdot \rho_y(a)] + \left( \sum_{a=15}^{49} [a \cdot \rho_t(a)] \right) \cdot y \right] \]  

(7.13)

Equation (7.13) shows that, under the assumptions stated above, \( \mu(f_y^c) \) is a linear function of \( y \).

Taking the first derivative with respect to \( y \) on both sides of equation (7.13), we get

\[ \left[ \mu(f_y^c) \right]' = \frac{1}{LFR} \sum_{a=15}^{49} [a \cdot \rho_t(a)] \]  

(7.14)

Equation (7.14) shows that, under the assumptions stated above, \( \left[ \mu(f_y^c) \right]' \) is constant with respect to \( y \) and implies that \( \sum_{a=15}^{49} [a \cdot \rho_t(a)] = LFR \cdot \left[ \mu(f_y^c) \right]' \). Consequently, equation (7.12) becomes

\[ TFR_t = LFR \cdot (1 - \left[ \mu(f_y^c) \right]') \]  

(7.15)

Equation (7.15) shows that, under the assumptions stated above, \( TFR_t \) is constant with respect to \( t \).

### 8. Effect of change in the cohort standard deviation on \( G_t \)

Mathematically, it is very complex to investigate, in a general way, the effect of change in the cohort standard deviation on \( G_t \). Therefore, we have to assume that the standardized cohort fertility schedule (i.e. \( h_t^c(a) \)) follow certain continuous probability distribution. For this purpose, we will use the following Gamma function for the simulation.

\[ g(a) = \begin{cases} 
K \cdot (a - a_0)^4 \cdot e^{-B (a - a_0)}, & \text{when } a \geq a_0 \\
0, & \text{when } a < a_0
\end{cases} \]  

(8.1)

The properties of the above-defined Gamma function are discussed in detail in Annex B.

In order to examine the effect of change (increment/decrement, denoted as \( \Delta \sigma(g) \)) in the cohort standard deviation on \( G_t \), we calculated the corresponding values of \( G_t \) using the above Gamma distribution. In this connection, three scenarios were simulated as follows:
Scenario 1: The mean age of fertility (\( \mu(g) \)) is held constant at 26 for all birth cohorts and the standard deviation of the benchmark cohort (i.e. the start cohort) is set to be 5.

Scenario 2: The mean age of fertility (\( \mu(g) \)) is held constant at 30 for all birth cohorts and the standard deviation of the benchmark cohort (i.e. the start cohort) is set to be 5.

Scenario 3: The mean age of fertility (\( \mu(g) \)) is held constant at 34 for all birth cohorts and the standard deviation of the benchmark cohort (i.e. the start cohort) is set to be 5.

The results of the numerical simulations are shown in Figure 10. From the results of the three scenarios, we notice that (i) if the change in the cohort standard deviation is positive, then \( G_t > 1 \); (ii) the higher the mean age of fertility, the larger the effect of change in the cohort standard deviation on \( G_t \) is. The numerical simulations also show that when \( 26 \leq \mu(g) \leq 34 \) and \(-0.10 \leq \Delta \sigma(g) \leq 0.10\), we have \( 0.99 < G_t < 1.03 \). Therefore, based on the numerical simulations, it is plausible to conclude that the effect of change in the cohort standard deviation on \( G_t \) is basically negligible. In terms of the shape of the curves depicted in Figure 10, they are close to parabolas on the interval \([-0.10, 0.10]\) of \( \Delta \sigma(g) \). If the following quadratic function

\[
G_t = 1 + \varphi_1 \cdot \Delta \sigma(g) + \varphi_2 \cdot (\Delta \sigma(g))^2
\]

where \( \varphi_1 \) and \( \varphi_2 \) are coefficients, is used to fit the curves, then we have \( R^2 > 0.992 \) (i.e. the coefficient of determination) for all the three scenarios.

**Figure 10. Effect of change in the cohort standard deviation on** \( G_t \)

![Figure 10](image_url)
Similarly, we simulated the effects of change in the cohort standard deviation on the standard deviation of \( H_p^t(a) \). Figure 11 shows the results for the three scenarios. It is clear that, when \( 26 \leq \mu(g) \leq 34 \) and \( -0.10 \leq \Delta \sigma(g) \leq 0.10 \), the standard deviation of \( H_p^t(a) \) is a monotonically increasing function of \( \Delta \sigma(g) \) when the cohort mean age of fertility (\( \mu(g) \)) is held constant. In terms of the shape of the curves depicted in Figure 11, they are close to straight lines on the interval \([-0.10, 0.10]\) of \( \Delta \sigma(g) \), with the coefficient of determination \( R^2 > 0.993 \) for all the three scenarios.

**Figure 11. Effect of change in the cohort standard deviation on the standard deviation of \( H_p^t(a) \)**

![Graph showing the effect of change in the cohort standard deviation on the standard deviation of \( H_p^t(a) \).](image)

9. **Suggestion for further study**

In this paper, the relationships between period and cohort fertility are examined in various ways from a cohort-to-period perspective. Similarly, the relationships could also be examined from a period-to-cohort perspective. To gain further insights into the relationships, more numerical simulations and empirical analyses could be conducted.
Annex A. Some properties of the (statistical) moments

In this annex, we discuss some properties of the (statistical) moments that are relevant to the present paper. Suppose that function \( p(x) \) represents a probability distribution (i.e. \( p(x) \) satisfies \( p(x) \geq 0 \) and \( \sum_{\text{all } x} p(x) = 1 \)), then its moments are defined as follows:

The \( r \)th absolute moment (about zero or origin) of \( p(x) \) is defined as
\[
\hat{M}_r(p) = \sum_{\text{all } x} [x^r \cdot p(x)]
\]  
(A.1)

where \( r \) is a non-negative integer. It is obvious that \( \hat{M}_0(p) = 1 \) and \( \hat{M}_1(p) = \mu(p) \), where \( \mu(p) = \sum_{\text{all } x} [x \cdot p(x)] \) is the mean of \( p(x) \).

The \( r \)th central moment (about mean) of \( p(x) \) is defined as
\[
\tilde{M}_r(p) = \sum_{\text{all } x} [(x - \mu(p))^r \cdot p(x)]
\]  
(A.2)

where \( r \) is a non-negative integer. It is obvious that \( \tilde{M}_0(p) = 1 \), \( \tilde{M}_1(p) = 0 \), and \( \tilde{M}_2(p) = \nu(p) \), where \( \nu(p) = \sum_{\text{all } x} [(x - \mu(p))^2 \cdot p(x)] \) is the variance of \( p(x) \).

The absolute and the central moments are important statistical measures for describing the position and the shape of a probability distribution.

By the binomial theorem, we have
\[
\hat{M}_r(p) = \sum_{\text{all } x} \left\{ \sum_{i=0}^{r} \left[ \frac{r!}{i!(r-i)!} \cdot x^{r-i} \cdot (-\hat{M}_i(p))^i \right] \cdot p(x) \right\}
\]

\[
= \sum_{i=0}^{r} \left\{ (-1)^i \left[ \frac{r!}{i!(r-i)!} \cdot (\hat{M}_i(p))^i \cdot \sum_{\text{all } x} [x^{r-i} \cdot p(x)] \right] \right\}
\]

\[
= \sum_{i=0}^{r} \left\{ (-1)^i \left[ \frac{r!}{i!(r-i)!} \cdot (\hat{M}_i(p))^i \cdot \hat{M}_{r-i}(p) \right] \right\}
\]  
(A.3)

Equation (A.3) gives the general relationship between the central and the absolute moments.
Furthermore, the skewness and the kurtosis of \( p(x) \) are defined as:

**Skewness:**
\[
\text{Skewness: } s(p) = \sum_{all \, x} \left( \frac{x - \mu(p)}{\sigma(p)} \right) \cdot p(x)
\]

**(A.4)**

**Kurtosis:**
\[
\text{Kurtosis: } k(p) = \sum_{all \, x} \left( \frac{x - \mu(p)}{\sigma(p)} \right)^4 \cdot p(x)
\]

**(A.5)**

where \( \sigma(p) = \sqrt{v(p)} \) is the standard deviation of \( p(x) \). It is obvious that

\[
\tilde{M}_3(p) = (\sigma(p))^3 \cdot s(p)
\]

**(A.6)**

\[
\tilde{M}_4(p) = (\sigma(p))^4 \cdot k(p)
\]

**(A.7)**

Based on the above definitions, we have

\[
\mu(p) = \tilde{M}_1(p)
\]

**(A.8)**

\[
v(p) = \tilde{M}_2(p) = \tilde{M}_2(p) - (\tilde{M}_1(p))^2
\]

**(A.9)**

\[
\sigma(p) = \sqrt{v(p)} = \sqrt{\tilde{M}_2(p) - (\tilde{M}_1(p))^2}
\]

**(A.10)**

\[
s(p) = \frac{\tilde{M}_3(p)}{(\sigma(p))^3} = \frac{1}{[\tilde{M}_2(p) - (\tilde{M}_1(p))^2]^{3/2}} \cdot [\tilde{M}_3(p) - 3 \cdot \tilde{M}_1(p) \cdot \tilde{M}_2(p) + 2 \cdot (\tilde{M}_1(p))^3]
\]

**(A.11)**

\[
k(p) = \frac{\tilde{M}_4(p)}{(\sigma(p))^4} = \frac{1}{[\tilde{M}_2(p) - (\tilde{M}_1(p))^2]^2}
\cdot [\tilde{M}_4(p) - 4 \cdot \tilde{M}_1(p) \cdot \tilde{M}_3(p) + 6 \cdot (\tilde{M}_1(p))^2 \cdot \tilde{M}_2(p) - 3 \cdot (\tilde{M}_1(p))^4]
\]

**(A.12)**

Since

\[
\tilde{M}_2(p) = \sum_{all \, x} [(x - \mu(p))^2 \cdot p(x)]
\]

\[
= \tilde{M}_2(p) - (\tilde{M}_1(p))^2
\]

**(A.13)**

\[
\tilde{M}_3(p) = \sum_{all \, x} [(x - \mu(p))^3 \cdot p(x)]
\]

\[
= \tilde{M}_3(p) - (\tilde{M}_1(p))^3 - 3 \cdot \tilde{M}_1(p) \cdot [\tilde{M}_2(p) - (\tilde{M}_1(p))^2]
\]

**(A.14)**

\[
\tilde{M}_4(p) = \sum_{all \, x} [(x - \mu(p))^4 \cdot p(x)]
\]

\[
= \tilde{M}_4(p) - (\tilde{M}_1(p))^4 - 4 \cdot \tilde{M}_1(p) \cdot [\tilde{M}_3(p) - (\tilde{M}_1(p))^3]
\]
\[ + 6 \cdot (\hat{M}_i(p))^2 \cdot [\hat{M}_2(p) - (\hat{M}_i(p))^2] \]  

we have

\[
\hat{D}_2(p) = \hat{M}_2(p) - (\hat{M}_i(p))^2
\]

\[
\hat{D}_3(p) = \hat{M}_3(p) - (\hat{M}_i(p))^3
\]

\[
\hat{D}_4(p) = \hat{M}_4(p) - (\hat{M}_i(p))^4
\]  

\( (A.15) \)
Annex B. Some properties of the Gamma function

The general formula for the Gamma function is given by

\[ g(a) = \begin{cases} K \cdot (a - a_0)^A \cdot e^{-B(a - a_0)}, & \text{when } a \geq a_0 \\ 0, & \text{when } a < a_0 \end{cases} \]  

(B.1)

where (i) \( a_0 \) is a constant, representing the start point of the curve, (ii) \( A (A > 0) \) and \( B (B > 0) \) are constants, which determine the shape of the curve, and (iii) \( K (K > 0) \) is a coefficient, which ensures that \( \int_{a_0}^{\infty} g(a) da = 1 \), i.e. \( K \cdot \int_{a_0}^{\infty} [(a - a_0)^A \cdot e^{-B(a - a_0)}] da = 1 \). Therefore, \( K = \frac{1}{\int_{a_0}^{\infty} [(a - a_0)^A \cdot e^{-B(a - a_0)}] da} \) once \( a_0, A \) and \( B \) are known. It is obvious that \( g(a) \geq 0, \ -\infty < a < \infty \).

The following graph shows two concrete examples of curve \( g(a) \): \( g_1(a) \) (\( a_0 = 15, \ \mu(g_1) = 29, \ \sigma(g_1) = 3 \)) and \( g_2(a) \) (\( a_0 = 15, \ \mu(g_2) = 29, \ \sigma(g_2) = 5 \)).

![Figure B.1. Examples of Gamma distribution](image_url)

The following graph shows in three-dimensional form the surface of \( g(a) \), using \( g_1(a) \) as the start curve and \( g_2(a) \) as the end curve. The curves between \( g_1(a) \) and \( g_2(a) \) are derived based on linear interpolation vis-à-vis the standard deviations.
The $n^{th}$ absolute moment (about zero or origin) of $g(a)$ is defined as $\hat{M}_n(g) = \int_{a_0}^{\infty} [a^n \cdot g(a)]da$, where $n$ is a non-negative integer. Let $u = a - a_0$, then we have $u \geq 0$, $a = u + a_0$, and $da = du$. Hence,

$$
\hat{M}_n(g) = \int_{a_0}^{\infty} [a^n \cdot g(a)]da = K \cdot \int_{0}^{\infty} [(u + a_0)^n \cdot u^A \cdot e^{-Bu}]du
$$

$$
= K \cdot \int_{0}^{\infty} \sum_{i=0}^{n} \left( \frac{n!}{i!(n-i)!} \cdot u^{n-i} \cdot a_0^i \right) \cdot u^A \cdot e^{-Bu} \]du
$$

$$
= K \cdot \sum_{i=0}^{n} \left[ a_0^i \cdot \frac{n!}{i!(n-i)!} \int_{0}^{\infty} (u^{A+(n-i)} \cdot e^{-Bu})du \right]
$$

Let $U_{n-i} = \int_{0}^{\infty} (u^{A+(n-i)} \cdot e^{-Bu})du$, then it is obvious that $K \cdot U_0 = K \cdot \int_{0}^{\infty} (u^A \cdot e^{-Bu})du = 1$ or $U_0 = 1/K$.

Further, we have

$$
U_{n-i} = -\frac{1}{B} \int_{0}^{\infty} u^{A+(n-i)} d(e^{-Bu}) = -\frac{1}{B} \left[ (u^{A+(n-i)} \cdot e^{-Bu}) \right]_{0}^{\infty} - \int_{0}^{\infty} e^{-Bu} d(u^{A+(n-i)})
$$

$$
= -\frac{1}{B} \left[ \lim_{u \to \infty} (u^{A+(n-i)} \cdot e^{-Bu}) - [A + (n-i)] \cdot \int_{0}^{\infty} (u^{A+(n-i)-1} \cdot e^{-Bu})du \right]
$$

$$
= -\frac{1}{B} \left[ \lim_{u \to \infty} (u^{A+(n-i)} \cdot e^{-Bu}) - [A + (n-i)] \cdot U_{(n-i)-1} \right]
$$

(B.3)
By Taylor series expansion, we have

\[ e^{Bu} = 1 + \frac{Bu}{1!} + \frac{(Bu)^2}{2!} + \frac{(Bu)^3}{3!} + \cdots + \frac{(Bu)^r}{r!} + \cdots \]  

(B.4)

Taking a positive integer \( m \), such that \( m > A + n \), then we have \( \frac{(Bu)^m}{m!} \) or equivalently

\[ \frac{1}{e^{Bu}} < \frac{m!}{(Bu)^m} \left( \frac{1}{m!} \right)^m \]  

Since \( 0 \leq i \leq n \), it follows that \( A \leq A + (n-i) \leq A + n \). Therefore,

\[ 0 < u^{A+(n-i)} \cdot e^{-Bu} = \frac{u^{A+(n-i)}}{e^{Bu}} < \left( \frac{m!}{B^m} \right) \left( \frac{1}{u^{m-(A+n)}} \right) \]  

(B.5)

Since \( \lim_{u \to \infty} \left[ \left( \frac{m!}{B^m} \right) \left( \frac{1}{u^{m-(A+n)}} \right) \right] = 0 \), it follows from the squeeze theorem of limit that

\[ \lim_{u \to \infty} (u^{A+(n-i)} \cdot e^{-Bu}) = 0, \quad i = 0, 1, 2, \ldots, n. \]  

Therefore, equation (B.3) becomes

\[ U_{n-i} = \frac{A+(n-i)}{B} \cdot U_{(n-i)-1}, \quad i = 0, 1, 2, \ldots, n-1 \]  

(B.6)

By mathematical induction, it can be proved that \( K \cdot U_{n-i} = \left[ \prod_{j=1}^{n-i} (A+j) \right] / B^{n-i} \), \( i = 0, 2, \ldots, n-1 \) (or equivalently, \( K \cdot U_r = \left[ \prod_{j=1}^{r} (A+j) \right] / B^r \)). Therefore, equation (B.2) can be rewritten as

\[ \hat{M}_n(g) = \sum_{i=0}^{n} \left[ a_i' \cdot \frac{n!}{i! (n-i)!} \cdot (K \cdot U_{n-i}) \right] \]  

(B.7)

Specially, we have

\[ \hat{M}_1(g) = \sum_{i=0}^{1} \left[ a_i' \cdot \frac{1!}{i! (1-i)!} \cdot (K \cdot U_{1-i}) \right] = a_0 \cdot (K \cdot U_0) + K \cdot U_1 \]

\[ = a_0 + \frac{A+1}{B} \]  

(B.8)

\[ \hat{M}_2(g) = \sum_{i=0}^{2} \left[ a_i' \cdot \frac{2!}{i! (2-i)!} \cdot (K \cdot U_{2-i}) \right] \]

\[ = a_0^2 \cdot (K \cdot U_0) + 2 \cdot a_0 \cdot (K \cdot U_1) + K \cdot U_2 \]
\[ a_0^2 + 2 \cdot a_0 \cdot \frac{A+1}{B} + \frac{(A+1) \cdot (A+2)}{B^2} \]  

(B.9)

\[ \hat{M}_3(g) = \sum_{i=0}^{3} a_i \cdot \frac{3!}{i!(3-i)!} \cdot (K \cdot U_{3-i}) \]

\[ = a_0^3 \cdot (K \cdot U_0) + 3 \cdot a_0^2 \cdot (K \cdot U_1) + 3 \cdot a_0 \cdot (K \cdot U_2) + K \cdot U_3 \]

\[ = a_0^3 + 3 \cdot a_0^2 \cdot (K \cdot U_1) + 3 \cdot a_0 \cdot (K \cdot U_2) + K \cdot U_3 \]  

(B.10)

\[ \hat{M}_4(g) = \sum_{i=0}^{4} a_i \cdot \frac{4!}{i!(4-i)!} \cdot (K \cdot U_{4-i}) \]

\[ = a_0^4 \cdot (K \cdot U_0) + 4 \cdot a_0^3 \cdot (K \cdot U_1) + 6 \cdot a_0^2 \cdot (K \cdot U_2) + 4 \cdot a_0 \cdot (K \cdot U_3) + K \cdot U_4 \]

\[ = a_0^4 + 4 \cdot a_0^3 \cdot (K \cdot U_1) + 6 \cdot a_0^2 \cdot (K \cdot U_2) + 4 \cdot a_0 \cdot (K \cdot U_3) + K \cdot U_4 \]  

(B.11)

Therefore, we have

(i) **Mean of** \( g(a) \)

\[ \mu(g) = \int_{a_0}^{\infty} [a \cdot g(a)] da = \hat{M}_1(g) = a_0 + \frac{A+1}{B} \]  

(B.12)

(ii) **Variance of** \( g(a) \)

\[ \nu(g) = \int_{a_0}^{\infty} [(a - \mu(g))^2 \cdot g(a)] da = \hat{M}_2(g) - (\hat{M}_1(g))^2 \]

\[ = \frac{(A+1) \cdot (A+2)}{B^2} - \left( \frac{A+1}{B} \right)^2 = \frac{A+1}{B^2} \]  

(B.13)

(iii) **Standard deviation of** \( g(a) \)

\[ \sigma(g) = \sqrt{\nu(g)} = \frac{\sqrt{A+1}}{B} \]  

(B.14)

(iv) **Skewness of** \( g(a) \)

\[ s(g) = \int_{a_0}^{\infty} \left( \frac{a - \mu(g)}{\sigma(g)} \right)^3 \cdot g(a) da \]
\[ = \frac{1}{(\sigma(g))^3} \cdot [\hat{M}_3(g) - 3 \cdot \hat{M}_1(g) \cdot \hat{M}_2(g) + 2 \cdot (\hat{M}_1(g))^3] \]

\[ = \frac{1}{(\sigma(g))^3} \cdot \left[ \frac{(A+1) \cdot (A+2) \cdot (A+3)}{B^3} - 3 \cdot \frac{(A+1)^2 \cdot (A+2)}{B^3} + 2 \cdot \left( \frac{A+1}{B} \right)^3 \right] \]

\[ = \frac{1}{(\sigma(g))^3} \cdot 2 \cdot \left( \frac{A+1}{B} \right)^3 = 2 \cdot \left( \frac{\sqrt{A+1}}{A+1} \right) \]  
\[ \text{(B.15)} \]

(note that \( \sigma(g) \cdot B = \sqrt{A+1} \), therefore \( (\sigma(g) \cdot B)^3 = (A+1)^{3/2} \))

(v) Kurtosis of \( g(a) \)

\[ k(g) = \int_{a_b} \left[ \left( \frac{a - \mu(g)}{\sigma(g)} \right)^4 \right] \cdot g(a) \, da \]

\[ = \frac{1}{(\sigma(g))^4} \cdot [\hat{M}_4(g) - 4 \cdot \hat{M}_1(g) \cdot \hat{M}_3(g) + 6 \cdot (\hat{M}_1(g))^2 \cdot \hat{M}_2(g) - 3 \cdot (\hat{M}_1(g))^4] \]

\[ = \frac{1}{(\sigma(g))^4} \cdot \left[ \frac{(A+1) \cdot (A+2) \cdot (A+3) \cdot (A+4)}{B^4} - 4 \cdot \frac{(A+1)^2 \cdot (A+2) \cdot (A+3)}{B^4} \right. \]

\[ + 6 \cdot \frac{(A+1)^3 \cdot (A+2)}{B^4} - 3 \cdot \left( \frac{A+1}{B} \right)^4 \right] \]

\[ = \frac{1}{(\sigma(g))^4} \cdot \left( \frac{A+1}{B^4} \right)^4 \cdot 3 \cdot (A+3) = 3 \cdot \left( \frac{A+3}{A+1} \right) \]  
\[ \text{(B.16)} \]

(note that \( \sigma(g) \cdot B = \sqrt{A+1} \), therefore \( (\sigma(g) \cdot B)^4 = (A+1)^2 \))

From equation (B.12), we have \( \frac{A+1}{B} = \mu(g) - a_0 \). Therefore, from equation (B.13), we have

\[ B = \left( \frac{A+1}{B} \right) / \nu(g) = \frac{\mu(g) - a_0}{\nu(g)} \]

\[ \text{(B.17)} \]

It then follows that

\[ A = B \cdot (\mu(g) - a_0) - 1 = \frac{(\mu(g) - a_0)^2}{\nu(g)} - 1 \]  
\[ \text{(B.18)} \]

The first derivative of \( g(a) \) with respect to \( a \) is as follows:
\[ [g(a)]_a = K \cdot (a - a_0)^{d-1} \cdot e^{-B \cdot (a - a_0)} \cdot [A - B \cdot (a - a_0)] \]  \hspace{1cm} (B.19)

By setting \([g(a)]_a = 0\), we obtain \(a = a_0 + \frac{A}{B}\) (ignore \(a = a_0\)). This tells us that the curve \(\gamma(a)\) attains its maximum at \(a_{\text{max}} = a_0 + \frac{A}{B}\). Since \(\mu(a) - a_{\text{max}} = 1/B > 0\), we have \(\mu(a) > a_{\text{max}}\). This implies that curve \(g(a)\) is always positively skewed (i.e. with the longer tail always on the right-hand side of the curve).

In terms of data fitting using the gamma function, the following method can be used. Taking the natural logarithm on both sides of equation (B.1), we obtain

\[ \ln[g(a)] = \ln(K) + A \cdot \ln(a - a_0) - B \cdot (a - a_0) \]  \hspace{1cm} (B.20)

where \(a > a_0\). Let \(y = \ln[g(a)]\), \(\theta_0 = \ln(K)\), \(\theta_1 = A\), \(\theta_2 = -B\), \(x_1 = \ln(a - a_0)\) and \(x_2 = a - a_0\), then equation (B.16) becomes:

\[ y = \theta_0 + \theta_1 \cdot x_1 + \theta_2 \cdot x_2 \]  \hspace{1cm} (B.21)

By applying bivariate linear regression to equation (B.18), we can obtain the estimates for coefficients \(\theta_0\), \(\theta_1\) and \(\theta_2\) (denoted as \(\hat{\theta}_0\), \(\hat{\theta}_1\) and \(\hat{\theta}_2\), respectively). Then we have \(K = e^{\hat{\theta}_0}\), \(A = \hat{\theta}_1\), and \(B = -\hat{\theta}_2\).
References


